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Novel indicators for identifying critical *INFRA*structure at *RISK* from Natural Hazards

Deliverable D6.3

Decision-Making Protocol

Primary Author	Pieter van Gelder, Noel van Erp/ Probabilistic Solutions Consult and Training (PSCT)
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Project Coordinator:	Professor Eugene O' Brien Roughan & O' Donovan Limited eugene.obrien@rod.ie
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Partners:

Roughan & O' Donovan Limited, Ireland

Eidgenössische Technische Hochschule Zürich, Switzerland.

Dragados SA, Spain.

Gavin and Doherty Geosolutions Ltd., Ireland.

Probabilistic Solutions Consult and Training BV, Netherlands.

Agencia Estatal Consejo Superior de Investigaciones Científicas, Spain.

University College London, United Kingdom.

PSJ, Netherlands.

Stiftelsen SINTEF, Norway.

Ritchey Consulting AB, Sweden.

University of Southampton (IT Innovation Centre), United Kingdom.

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Executive Summary

Probability theory has as its subject matter the assignment of quantitive measures of plausibility (i.e. probabilities) to logical propositions like, for example, “this bridge will fail” or “this bridge will not fail”. Probability theory allows us to compute outcome probability distributions for a given (stress) scenario. Outcome probability distributions are the information carriers of our probability theoretical analyses as they (1) enumerate all the possible outcomes under a given scenario/decision and (2) assign probabilities to each of these outcomes.

Decision theory has as its subject matter the choosing between alternative actions. Each course of action will lead us to some state of the world which corresponds with some outcome probability distribution. So (1) once we have enumerated all the viable courses of actions, we (2) may proceed to construct the outcome probability distributions that correspond with each of these actions, and (3) based on the maximization of some measure which is defined on these outcome probability distributions choose the optimal course of action.

The most well-known decision theories are the expected outcome theory and the expected utility theory. Elements of these decision theories are used as a matter of course in probabilistic cost-benefit analyses. This despite the fact that the behavioural economics community, consisting of economists and experimental psychologists, have declared these decision theories to be fundamentally flawed, as they point to psychological experiments that would seem to indicate that human decision making does not adhere to the maximization of expectation values of either outcome probability distributions or utility probability distributions.

In this deliverable a new decision theory is introduced that resolves this apparent inconsistency between probabilistic cost-benefit practice and the findings of the behavioural economists. The Bayesian decision theory is neo-Bernoullian in that it rederives Bernoulli’s utlity function by way of a consistency proof. But it differs from Bernoulli’s original utility in that it proposes that the expectation value indeed need not be the most appropriate criterion of choice for our actions.

First we will apply the Bayesian decision theory in order to evaluate the effects of the stress scenarios in transport systems example which was discussed in D6.2. We then will work out its practical implication, by comparing the investment willingness in hazard prevention measures under the different decision theories. It will also be discussed how the Bayesian decision theory may resolve some of the decision theoretical inconsistencies, which then will lead us to a principled recommendation for the computation of disproportionality factors of the cost-benefit methodology.

The research results of this deliverable are relevant for all those, be they infrastructural managers or not, that wish to make a probabilistic cost-benefit analysis.

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1.0 INTRODUCTION

We recommend the Bayesian decision theory as the decision-making protocol of the stress test framework because the decision theoretical inconsistencies of the expectation value as a measure of risk is so problematic – in that predicted decisions do not correspond with observed decisions – that it has given rise to the paradigm of behavioral economics[1]. The Bayesian decision theory proposes as a solution to these inconsistencies (1) a re-instatement of Bernoulli's original utility function, and (2) the use of an alternative measure of risk.

First we will apply the Bayesian decision theory in order to evaluate the effects of the stress scenarios in transport systems example which was discussed in D6.2 (van Erp et al., 2016), in order to give the reader a general sense of the decision making protocol that is proposed in D6.3.

Then we will proceed to formally derive the Bayesian decision theory. In the Bayesian decision theory irisk is operationalized as some position measure on an outcome probability distribution. As we have to navigate our action space, then we will typically choose that action that maximizes the position of a given outcome probability distribution (if gains are represented by positive numbers and losses by negative numbers, that is). Also, it is found that the conversion of objective monies to their subjective worth (as the worth of a given sum of money may differ from one person to another, depending on their current asset position), is best done by way of Bernoulli's utility function, as this utility function (together with Stevens' power law) is the only consistent utility function.

We then work out the practical implications of the proposed Bayesian decision theory, by comparing the investment willingness in hazard prevention measures under the proposed Bayesian decision theory with the investment willingness under the expected outcome theory and expected utility theory. The decision theoretical scenario chosen for this toy problem is based on investment choices Dutch policy makers were faced with following the great Dutch flooding in 1953. This discussion will lead us to a practical proposal on how to model the Disproportionality Factors of cost-benefit analyses.

Finally, it will be discussed how the Bayesian decision theory may resolve some of the decision theoretical inconsistencies, as we discuss the rational basis for the fact that our common sense strongly suggests High Impact Low Probability (HILP) events in some sense are much 'riskier' than Low Impact High Probability (LIHP) events, even if the probability times consequence summation gives us the same expectation values for both types of events.

[1] Behavioral economics is a multidisciplinary field which combines experimental psychology with economics and which has as one of its central tenets that human decision making cannot be captured by simple mathematical maximization principles.

2.0 EVALUATING THE EFFECT OF STRESS SCENARIOS IN TRANSPORT SYSTEMS

In this chapter we will apply the decision making protocol to the simple bridge system example which is given in D6.2 (van Erp et al., 2016). In this example a stress test involving increased river discharge values is performed for a 5-bridge system which consists of two types of bridges. We will use the decision making protocol to guide us in our decision making as to which components to strengthen in the hypothetical bridge system.

2.1 Risk Definition

As already stated in D6.2, we assume that risk is some function of both consequences, $\mathbf{x} = \{x_1, \ldots, x_n\}$, and the probabilities of these consequences, $\mathbf{p} = \{p_1, \ldots, p_n\}$;, that is,

$$\text{Risk} = f(\mathbf{x}, \mathbf{p}). \tag{2.1}$$

Now, it will be argued in Chapter 3 that the risk function $f(\mathbf{x}, \mathbf{p})$ may be interpreted as a position measure on the corresponding outcome probability distribution:

$$p(\mathbf{x}) = \begin{cases} p_1, & x_1, \\ p_2, & x_2, \\ \vdots & \\ p_n, & x_n. \end{cases} \tag{2.2}$$

where the $\mathbf{x} = \{x_1, \ldots, x_n\}$ are mapped on the *x*-axis and the $\mathbf{p} = \{p_1, \ldots, p_n\}$ are mapped on the *y*-axis. For example, if we take as our risk function $f(\mathbf{x}, \mathbf{p})$ the expectation value:

$$f(\mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} x_i \, p_i = E(X), \tag{2.3}$$

then we have that our risk index is a measure of the position of the most-likely scenario (of losses). Now, given the ubiquitousness of (2.3) as a definition for risk, there must be some merit in taking the most-likely loss-scenario as our risk measure.

An alternative, more cautious position is taken by the return period methodology, which takes as its risk index the measure the position of an unlikely (to be on the safe side of things) worst-case scenario of some hazard intensity magnitude:

$$f(\mathbf{x}, \mathbf{p}) = E(X) + k \, std(X), \tag{2.4}$$

where

$$E(X) = \sum_{i=1}^{n} x_i \, p_i, \qquad std(X) = \sqrt{\sum_{i=1}^{n} x_i^2 \, p_i - \left(\sum_{i=1}^{n} x_i \, p_i \right)^2}, \tag{2.5}$$

and k is the sigma-level that will give us the desired upper percentile value.

In Chapter 3 it is proposed that the position measure that takes into account the worst, most-likely, and best case scenarios:

$$f(\mathbf{x}, \mathbf{p}) = \frac{LB(X) + E(X) + UB(X)}{3}. \tag{2.6}$$

Now, there are as of yet no guiding mathematical principles by which to choose between the alternative risk indices (20.3), (2.4), and (2.6). We have only common sense principles like those expounded in Chapter 3 to guide us, when it comes to this decision theoretical degree of freedom (van Erp et al., 2016*a*).

2.2 Risk Minimisation

An infrastructure stress test is an analysis conducted under unfavorable scenarios which is designed to determine whether there are unacceptable infrastructure related risks. These tests are meant to detect objects that if “strengthened” through the execution of preventive interventions will greatly decrease the infrastructure related risk (Adey, 2016). So the strengthening of one or more infrastructural objects are the alternative actions which are open to the road manager, relative to a status quo where he only performs regular maintenance. Moreover, the road manager will have some budget constraint under which he has to decide whether or not to strengthen additional infrastructural objects or not.

If we enumerate all the possible actions that a road manager might take as the set

$$\{A_1, A_2, \ldots, A_m\}. \tag{2.7}$$

Then we have, for a given stress scenario S, that each action A_k will map to a specific conditional outcome probability distribution:

$$p(\mathbf{x} \mid S, A_k) = \begin{cases} p_1, & x_1, \\ p_2, & x_2, \\ \vdots & \\ p_{n_k}, & x_{n_k}. \end{cases} \tag{2.8}$$

where n_k is the number of possible outcomes under the *k*th action A_k. Now, we may compute for each of these outcome probability distributions (10.8), depending on our risk appetite, any of the risk indices (2.3), (2.4), and (2.6). For a given choice of risk index, the road manager then chooses that decision

$$A_k \in \{A_1, A_2, \ldots, A_m\} \tag{2.9}$$

which has the lowest risk (index) value of its corresponding outcome probability distribution (2.8).

So, for a given stress test scenario S, in this straightforward decision theoretical approach (Jaynes, 2003), all that needs to be done is:

1. An enumeration of all the possible action (2.7),
2. The construction of the corresponding outcome probability distributions (2.8), conditional on the stress scenario S,
3. A commitment to one of the risk indices either (2.3), (2.4), or (2.6),
4. A minimization of the chosen risk index over the set of possible actions (2.9).
5.

For an actual demonstration of this approach, see the following section, as well as Chapter 4.

2.3 Applying the Decision-Making Protocol

In Chapter 3 of D6.2 a simple example is given of how a stress test may be done a simple system of bridges. We again consider this road network with 5 components (bridges over a river), Figure 2.1.

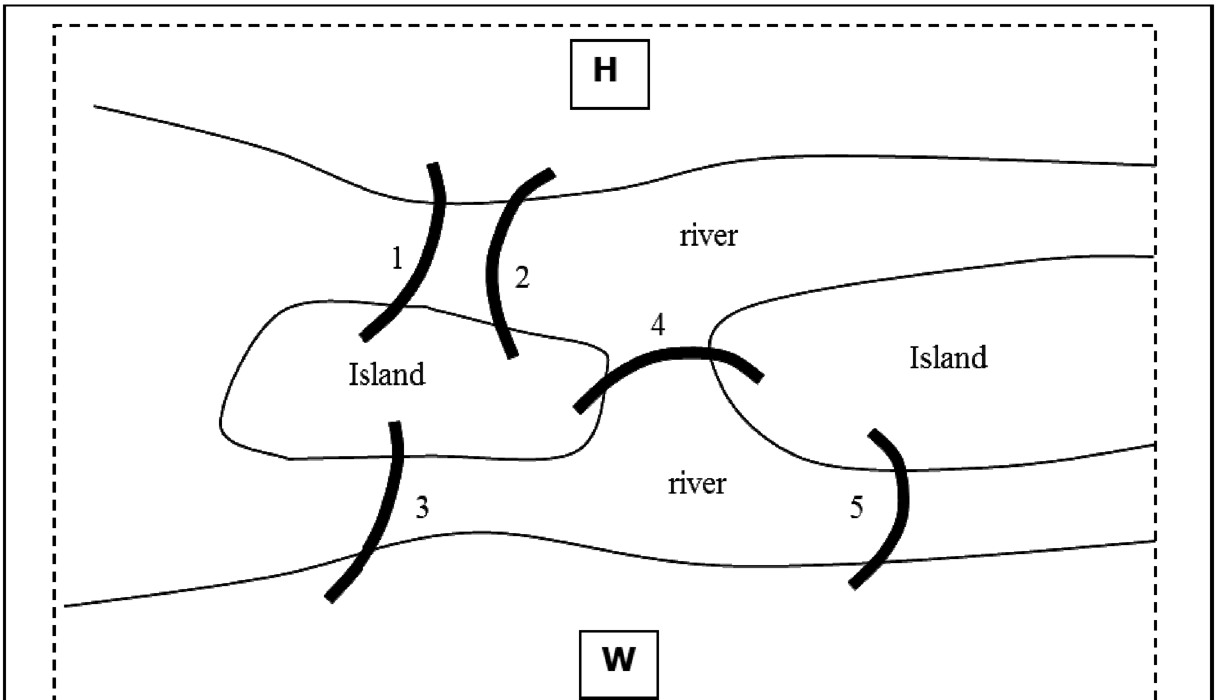

Figure 2.1: Bridge System

We have that bridges 1, 2, and 3 are bridges of type I and bridges 4 and 5 of type II. Each type of bridge has its own characteristics and, as a consequence, will behave differently under different scour conditions; that is, bridges of type I are considered to be more resistant to scour than those of type II. The scour load for a given bridge is considered to be some limit state function of some discharge value Q measured in the vicinity of the bridge (e.g. upstream, downstream, etc.).In what follows we will assume some, say, precipitation stress scenario that will lead to elevated flood discharges throughout the river; that is, elevated flood discharges are predicted in the vicinity of the bridges as in Figure 2.2.

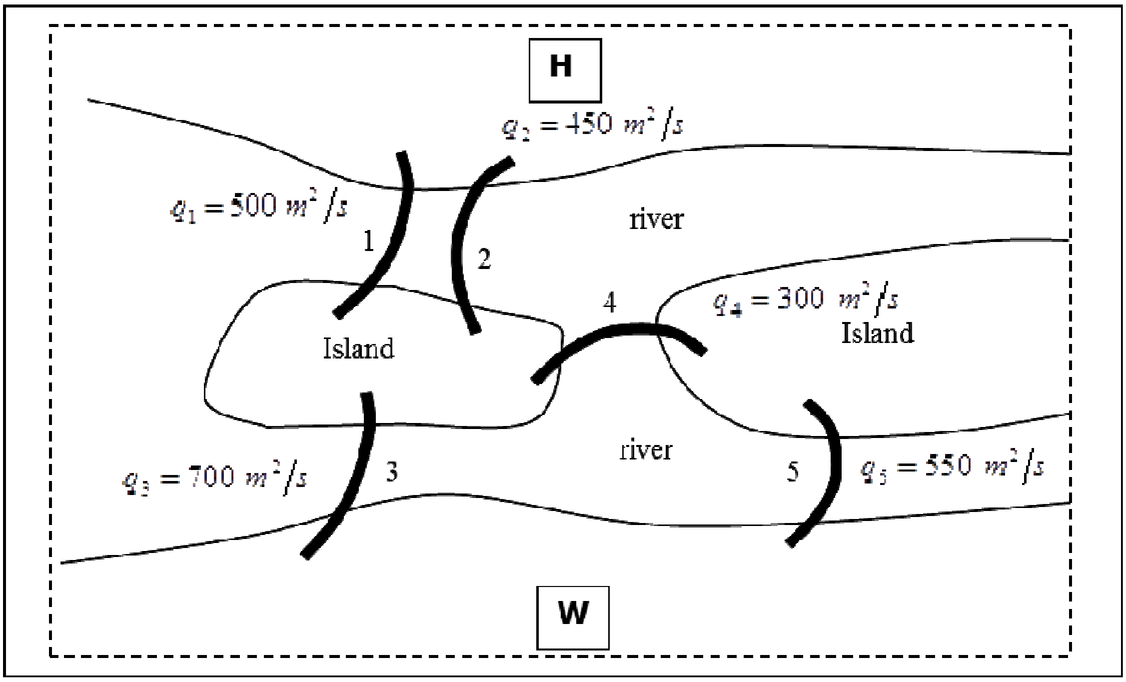

Figure 2.2: River Discharge Stress Scenario

For both bridge types in Figure 2.2, the probability of being in damage state $i = 0$ (undamaged state) is modelled as

$$\pi(i = 0 \mid \alpha_1, \beta, Q) = 1 - \Phi\left[\frac{\ln(Q/\alpha_1)}{\beta}\right]; \qquad (2.10a)$$

the probability of being in damage state $i = 1$ is a modelled as

$$\pi(i = 1 \mid \alpha_1, \alpha_2, \beta, Q) = \Phi\left[\frac{\ln(Q/\alpha_1)}{\beta}\right] - \Phi\left[\frac{\ln(Q/\alpha_2)}{\beta}\right]; \qquad (2.10b)$$

the probability of being in damage state $i = 2$ is

$$\pi(i = 2 \mid \alpha_2, \alpha_3, \beta, Q) = \Phi\left[\frac{\ln(Q/\alpha_2)}{\beta}\right] - \Phi\left[\frac{\ln(Q/\alpha_3)}{\beta}\right]; \qquad (2.10c)$$

the probability of being in damage state $i = 3$ is

$$\pi(i = 3 \mid \alpha_3, \beta, Q) = \Phi\left[\frac{\ln(Q/\alpha_3)}{\beta}\right], \qquad (2.10d)$$

The actual parameters α_1, α_2, α_3, and β in (2.10) are the fragility parameters of the fragility curves

$$P(i \mid \alpha_i, \beta, Q) = \Phi\left[\frac{\ln(Q/\alpha_i)}{\beta}\right], \qquad \text{for } i = 1,2,3, \qquad (2.11)$$

where it is understood that the fragility parameters will differ for each type of bridge.

2.3.1 Outcome Probability Distribution Under Decision To Do Nothing

Let us assume that for the bridge of type I we have 10 discharge values Q_j, for $j = 1,\ldots,10$. Let us also assume that we sample the limit state function for each discharge value and each damage state, starting from damage state $i = 1$, $N = 100$ times in order to determine each time the number of realizations Z_{ij} that are in the pertinent damage states $i = 1$, $i = 2$, and $i = 3$, respectively. In Table 2.1 we give a possible realization of such a sampling exercise.

	Q_j	Z_{1j}	Z_{2j}	Z_{3j}
j = 1	10	0	0	0
j =2	39	10	0	0
j =3	78	30	3	0
j =4	156	60	10	0
j =5	312	100	30	3
j =6	625	100	60	10
j =7	1250	100	100	30
j =8	2500	100	100	60
j =9	5000	100	100	100
j =10	10000	100	100	100

Table 2.1: Type I Bridge (discharge values and associated number of damage state realisations)

Based on the data in Table 2.1, we can specify the fragility-parameter likelihood model (Shinozuka et al., 2003):

$$L(\alpha_1,\alpha_2,\alpha_3,\beta)=\prod_{i=1}^{3}\prod_{j=1}^{10}\Phi\left[\frac{\ln(Q_j/\alpha_i)}{\beta}\right]^{Z_{ij}}\left(1-\Phi\left[\frac{\ln(Q_j/\alpha_i)}{\beta}\right]\right)^{N-Z_{ij}}, \quad (2.12)$$

where Φ is the symbol of the cumulative standard normal distribution. If we assign the following non-informative prior to the fragility-parameters (Jaynes, 1968)

$$p(\alpha_1,\alpha_2,\alpha_3,\beta)\propto\frac{1}{\alpha_1\alpha_2\alpha_3\beta}, \quad (2.13)$$

Then we may combine (2.12) and (2.12) into the posterior probability distribution (Jaynes, 2003)

$$p(\alpha_1,\alpha_2,\alpha_3,\beta\,|\,D)\propto\frac{1}{\alpha_1\alpha_2\alpha_3\beta}\prod_{i=1}^{3}\prod_{j=1}^{10}\Phi\left[\frac{\ln(Q_j/\alpha_i)}{\beta}\right]^{Z_{ij}}\left(1-\Phi\left[\frac{\ln(Q_j/\alpha_i)}{\beta}\right]\right)^{N-Z_{ij}} \quad (2.14)$$

where the data D consists of the set of inputted flow discharges $\{Q_j\}$ and the set of observed number of failure realisations $\{Z_{ij}\}$ in $N=100$ trials, for $i=1,2,3$ and $j=1,\ldots,10$, as shown in Table 2.1 and Table 2.2.

By way of the Nested Sampling algorithm (see D6.2, Chapter 6), we may obtain a univariate representation for the fragility parameter probability distribution (2.14) which allows us to evaluate the mean and standard deviation vectors:

$$\mu_{\alpha_1}=107.99,\quad \mu_{\alpha_2}=421.62,\quad \mu_{\alpha_3}=1651.18,\quad \mu_b=0.7008, \quad (2.15a)$$

$$\sigma_{\alpha_1}=5.81,\quad \sigma_{\alpha_2}=21.54,\quad \sigma_{\alpha_3}=66.11,\quad \sigma_b=0.0273. \quad (2.15b)$$

As the probabability distribution (2.14) cannot be easily factorized in the product of four independent probability dsitributions, one will need to use the univariate Nested Sampling representation of (2.14), say,

$$p_{NS}(\alpha_1,a_2,\alpha_3,\beta\,|\,D_1,\text{Type I},A_1), \quad (2.16)$$

where D_1 is as in Table 2.1 and (2.16) itself is a collection of probability weighted fragility parameter vectors, in order to take into account the fragility parameter uncertainty and A_1 is the 'action' to keep the status quo.

Now, in our hypothetical stress test problem of D6.2, we have that the type II are more vulnerable to scour. For the bridge of type II, we use the same 10 discharge values Q_j that were used in Table 2.1, where we sample from the same limit state function for each discharge value and each damage state, in order to determine each time the number of realizations Z_{ij} that are in the pertinent damage states $i=1$, $i=2$, and $i=3$, respectively. In Table 3.2 we give a possible realization of such a sampling exercise.

	Q_j	Z_{1j}	Z_{2j}	Z_{3j}
j = 1	10	10	0	0
j =2	39	30	3	0
j =3	78	60	10	0
j =4	156	100	30	3
j =5	312	100	60	10
j =6	625	100	100	30
j =7	1250	100	100	60
j =8	2500	100	100	100
j =9	5000	100	100	100
j =10	10000	100	100	100

Table 2.2: Type II Bridge (discharge values and associated number of damage state realisations)

Note that the difference in the number of Z_{ij} realizations, relative to Table 3.1, are due to the fact that the damage state model for a type II bridge will set all the damage state thresholds lower, as these types of bridges are more vulnerable to scour loading.

By way of the Nested Sampling algorithm (see D6.2, Chapter 6), we may obtain a univariate representation for the fragility parameter probability distribution (2.14) which allows us to evaluate the mean and standard deviation vectors, and the correlation-matrices of the fragility parameters:

$$\mu_{\alpha_1}=48.73, \quad \mu_{\alpha_2}=210.01, \quad \mu_{\alpha_3}=861.58, \quad \mu_b=0.7626, \tag{2.17a}$$

$$\sigma_{\alpha_1}=2.88, \quad \sigma_{\alpha_2}=11.80, \quad \sigma_{\alpha_3}=44.91, \quad \sigma_b=0.0323, \tag{2.17b}$$

As the probabability distribution (2.14) cannot be easily factorized in the product of four independent probability dsitributions, one will need to use the univariate Nested Sampling representation of (2.14), say,

$$p_{NS}\left(\alpha_1,a_2,\alpha_3,\beta \mid D_2,\text{Type II},A_1\right), \tag{2.18}$$

where D_2 is as in Table 2.2 and (2.18) itself is a collection of probability weighted fragility parameter vectors, in order to take into account the fragility parameter uncertainty, and A_1 is the action to keep the status quo.

Using the Nested Sampling proxies (2.16) and (2.18), we may take into account, by way of the Law of Total Probability and the fragility parameter uncertainty in (2.10):

$$\pi(i \mid Q, D_1, \text{Type I}, A_1) = \sum_{(\alpha_1, a_2, \alpha_3, \beta)} \pi(i, \alpha_1, a_2, \alpha_3, \beta \mid Q, D_1, \text{Type I}, A_1)$$

$$= \sum_{(\alpha_1, a_2, \alpha_3, \beta)} \pi(i \mid \alpha_1, \alpha_2, \alpha_3, \beta, Q)\; p_{NS}(\alpha_1, a_2, \alpha_3, \beta \mid D_1, \text{Type I}, A_1)$$

(2.19*a*)

and, likewise,

$$\pi(i \mid Q, D_2, \text{Type II}, A_1) = \sum_{(\alpha_1, a_2, \alpha_3, \beta)} \pi(i, \alpha_1, a_2, \alpha_3, \beta \mid Q, D_2, \text{Type II}, A_1)$$

$$= \sum_{(\alpha_1, a_2, \alpha_3, \beta)} \pi(i \mid \alpha_1, \alpha_2, \alpha_3, \beta, Q)\; p_{NS}(\alpha_1, a_2, \alpha_3, \beta \mid D_2, \text{Type II}, A_1)$$

(2.19*b*)

The fragility parameter weighted damage state probabilities for the type I bridges are given as, Figure 2.2 and (2.19*a*),

$$\pi(i \mid q_1 = 500, D_1, \text{Type I}, A_1) = [0.0147 \quad 0.3885 \quad 0.5522 \quad 0.0446], \qquad (2.20a)$$
$$\pi(i \mid q_2 = 450, D_1, \text{Type I}, A_1) = [0.0212 \quad 0.4410 \quad 0.5056 \quad 0.0322], \qquad (2.20b)$$
$$\pi(i \mid q_3 = 700, D_1, \text{Type I}, A_1) = [0.0040 \quad 0.2304 \quad 0.6547 \quad 0.1109], \qquad (2.20c)$$

and the fragility parameter weighted damage state probabilities for the type II bridges are given as, Figure 2.2 and (2.19*b*),

$$\pi(i \mid q_4 = 300, D_2, \text{Type II}, A_1) = [0.0089 \quad 0.3106 \quad 0.5966 \quad 0.0840], \qquad (2.20d)$$
$$\pi(i \mid q_5 = 550, D_2, \text{Type II}, A_1) = [0.0008 \quad 0.1027 \quad 0.6176 \quad 0.2789], \qquad (2.20e)$$

where the damage state probabilities are ordered as $i = 0,1,2,3$; that is, for the stress scenario that gives us river discharge values as in Figure 2.2, all the bridges have the largest probability to be in damage state 2. The resulting probability map is given in Table 2.3.

	i = 0	*i* = 1	*i* = 2	*i* = 3
Bridge 1	0.0147	0.3885	0.5522	0.0446
Bridge 2	0.0212	0.4410	0.5056	0.0322
Bridge 3	0.0040	0.2304	0.6547	0.1109
Bridge 4	0.0089	0.3106	0.5966	0.0840
Bridge 5	0.0008	0.1027	0.6176	0.2789

Table 2.3: Status Quo Damage State Probability Map of Bridge System under Stress Scenario

The repair costs of each of these damage states are assumed to be as in Table 2.4.

	$i = 0$	$i = 1$	$i = 2$	$i = 3$
Bridge 1	0	10.000	50.000	1.000.000
Bridge 2	0	10.000	50.000	1.000.000
Bridge 3	0	10.000	50.000	1.000.000
Bridge 4	0	6.000	24.000	480.000
Bridge 5	0	6.000	24.000	480.000

Table 2.4: Status Quo Damage State Repair Cost Map of Bridge System under Stress Scenario

The systems probability and consequence maps in Tables 2.3 and 2.4 translate for 1.000.000 Monte Carlo (MC) samples to the conditional outcome frequency distribution in Figure 2.3.

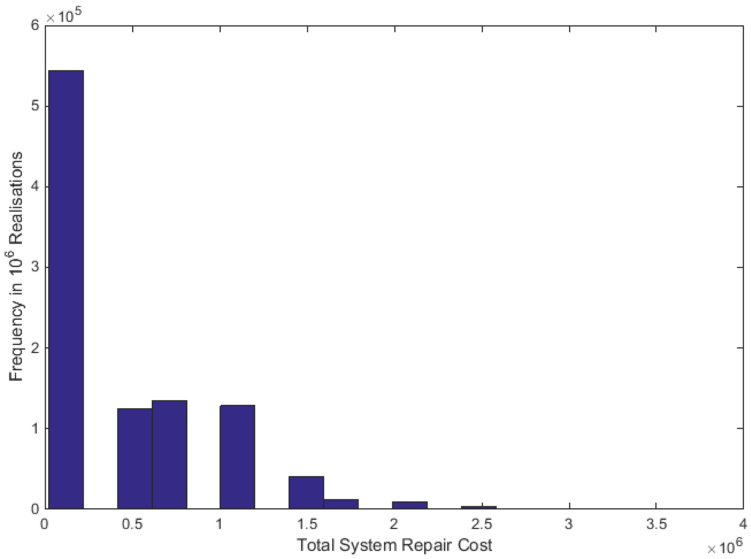

Figure 2.3: Frequency Distribution of Total Repair Costs under the Stress Scenario in Figure 2.2

The frequency distribution in Figure 2.3 has a mean and standard deviation of

$$\overline{X}_{total} = 488960, \qquad S_{total} = 468900. \qquad (2.21)$$

So under the hypothetical current status quo (i.e. the do-nothing action A_1) the stress test output is as in Figure 2.3.

2.3.2 Outcome Probability Distribution Under Decision To Improve Type I Bridges

Let us assume that for the bridge of type I the road manager has some protective measures in mind that is expected to reduce the vulnerability for scour for bridges of this type. Adjusting the limit state and damage threshold functions of the type I bridge and re- sampling (relative to the data in Table 2.1) the limit state function for each discharge value and each damage state, we obtain for each discharge value Q_j for each of the actual damage states (i.e., $i=1$, $i=2$, and $i=3$) the number of realizations Z_{ij} from a trial of $N=100$ replications that are in that damage state, Table 2.5.

	Q_j	Z_{1j}	Z_{2j}	Z_{3j}
j = 1	10	0	0	0
j =2	39	0	0	0
j =3	78	0	0	0
j =4	156	10	0	0
j =5	312	30	3	0
j =6	625	60	10	0
j =7	1250	100	30	3
j =8	2500	100	60	10
j =9	5000	100	100	30
j =10	10000	100	100	60

Table 2.5: River Discharge-Damage Realisations Data for Protected Type I Bridge

Note that the difference in the number of Z_{ij} realizations in Table 3.4 represent a "pushing to the right" (with two rows downwards) of the limit state sampled damage state threshold values relative to the damage state realizations in Table 2.1.

By way of the Nested Sampling algorithm (see D6.2, Chapter 6), we may obtain a univariate representation for the fragility parameter probability distribution (2.14) which allows us to evaluate the mean and standard deviation vectors:

$$\mu_{\alpha_1}=438.24, \quad \mu_{\alpha_2}=1801.19, \quad \mu_{\alpha_3}=7530.27, \quad \mu_b=0.8047, \tag{2.22a}$$

$$\sigma_{\alpha_1}=21.02, \quad \sigma_{\alpha_2}=63.08, \quad \sigma_{\alpha_3}=325.82, \quad \sigma_b=0.0136. \tag{2.22b}$$

As the probabability distribution (2.14) cannot be easily factorized in the product of four independent probability dsitributions, one will need to use the univariate Nested Sampling representation of (2.14), say,

$$p_{NS}\left(\alpha_1, a_2, \alpha_3, \beta \mid D_1, \text{Type I}, A_2\right), \tag{2.23}$$

where D_1 is as in Table 2.5 and (2.23) itself is a collection of probability weighted fragility parameter vectors, in order to take into account the fragility parameter uncertainty. Using the Nested Sampling proxy (2.23), we may take into account, by way of the Law of Total Probability and the fragility parameter uncertainty in (2.10):

$$\pi(i \mid Q, D_1, \text{Type I}, A_2) = \sum_{(\alpha_1, a_2, \alpha_3, \beta)} \pi(i, \alpha_1, a_2, \alpha_3, \beta \mid Q, D_1, \text{Type I}, A_2)$$

$$= \sum_{(\alpha_1, a_2, \alpha_3, \beta)} \pi(i \mid \alpha_1, \alpha_2, \alpha_3, \beta, Q)\ p_{NS}(\alpha_1, a_2, \alpha_3, \beta \mid D_1, \text{Type I}, A_2)$$

(2.24)

The fragility parameter weighted damage state probabilities for the type I bridges under the protection action A_1 are given as, Figure 2.2 and (2.24),

$$\pi(i \mid q_1 = 500, D_1, \text{Type I}, A_2) = [0.4345 \quad 0.5096 \quad 0.0555 \quad 0.0004], \qquad (2.25a)$$
$$\pi(i \mid q_2 = 450, D_1, \text{Type I}, A_2) = [0.4863 \quad 0.4710 \quad 0.0424 \quad 0.0002], \qquad (2.25b)$$
$$\pi(i \mid q_3 = 700, D_1, \text{Type I}, A_2) = [0.2810 \quad 0.5994 \quad 0.1188 \quad 0.0016], \qquad (2.25c)$$

and the fragility parameter weighted damage state probabilities for the type II bridges are as in (2.20*d*) and (2.20*e*),

$$\pi(i \mid q_4 = 300, D_2, \text{Type II}, A_2) = \pi(i \mid q_4 = 300, D_2, \text{Type II}, A_1), \qquad (2.25d)$$
$$\pi(i \mid q_5 = 550, D_2, \text{Type II}, A_2) = \pi(i \mid q_5 = 550, D_2, \text{Type II}, A_1), \qquad (2.25e)$$

where the damage state probabilities are ordered as $i = 0,1,2,3$. The resulting probability map is given in Table 2.6.

	i = 0	*i* = 1	*i* = 2	*i* = 3
Bridge 1	0.4345	0.5096	0.0555	0.0004
Bridge 2	0.4863	0.4710	0.0424	0.0002
Bridge 3	0.2810	0.5994	0.1188	0.0016
Bridge 4	0.0089	0.3106	0.5966	0.0840
Bridge 5	0.0008	0.1027	0.6176	0.2789

Table 2.6: Damage State Probability Map of Protected Bridge System under Stress Scenario

The repair costs of each of these damage states are assumed to be the same as those in Table 2.4, but the certain investment cost to implement the protection measures is

$$I = 275.000. \qquad (2.26)$$

For example, the repair cost of the damage state where the bridges 1, 2, and 3 are in damage state $i = 2$ and the bridges 4 and 5 are in damage state $i = 3$, that is,

$$\mathbf{x}^{(1)} = (2 \quad 2 \quad 2 \quad 3 \quad 3),$$

is found by taking the sum of the repair costs in Table 2.4 and the certain investment cost (2.26):

$$c\left(\mathbf{x}^{(1)} \mid A_2\right) = 50.000 + 50.000 + 50.000 + 480.000 + 480.000 + 275.000$$

$$= 1.385.000. \qquad (2.27a)$$

Alternatively, the system state where all the bridges are in the undamaged state $i = 0$,

$$\mathbf{x}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

now has a cost of, Table 2.4 and (2.26)

$$c\left(\mathbf{x}^{(2)} \mid A_2\right) = 0 + 0 + 0 + 0 + 0 + 275.000$$

$$= 275.000. \qquad (2.27b)$$

The systems probability and consequence maps in Tables 2.6 and 2.4 together with the certain investment cost (2.26) translate for 1.000.000 Monte Carlo (MC) samples to the conditional outcome frequency distribution in Figure 2.4[2].

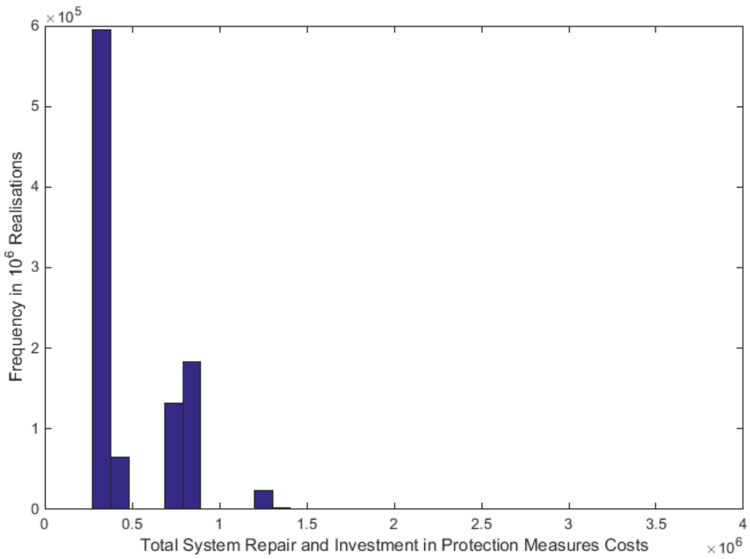

Figure 2.4: Frequency Distribution of Total Repair Costs for Protected System under Stress Scenario

The frequency distribution in Figure 2.4 has a mean and standard deviation of

$$\overline{X}_{total} = 509450, \qquad S_{total} = 247880. \qquad (2.28)$$

[2] The histogram bars in Figure 2.4 are less wide than in Figure 2.3, as both are histograms with 20 bars, but the original *x*-axis of Figure stopped at 2.5×10^6, by forcing this axis to be of the same length as the *x*-axis in Figure 2.3, the widths of histogram bars have changed relative to each other.

So under the protection action in which type I bridges are strengthened, the stress test output is as in Figure 2.4.

2.3.3 Choosing Between Outcome Probability Distributions

A quick inspection of the means and standard deviations in (2.21) and (2.28) shows that the under the stauts quo the projected costs will be slightly lower than under the investment in the strengthening of the type I bridges:

$$\overline{X}_{total} = 488960 \qquad \text{vs.} \qquad \overline{X}_{total} = 509450 . \tag{2.29}$$

However, the investment in protection measures does reduce the spread (i.e. volatility) in risk:

$$S_{total} = 468900 \qquad \text{vs.} \qquad S_{total} = 247880 . \tag{2.30}$$

Under the expected outcome/utility theories (2.3), we will neglect (2.30) as we choose for the action A_1 (i.e. do-nothing) , based on (2.29):

$$E(X \mid A_1) = 488960 < 509450 = E(X \mid A_2) . \tag{2.31}$$

In contrast, under the design return period approach (2.4), we will take (2.30) into account as we choose for the action A_2 (i.e. invest in protection measures) , based on the worst-case assessment (2.29) and (2.30):

$$\begin{aligned} E(X \mid A_1) + k \ \mathrm{std}(X \mid A_1) &= 488960 + k(468900) \\ &> 509450 + k(247880) \\ &= E(X \mid A_2) + k \ \mathrm{std}(X \mid A_2) \end{aligned} \tag{2.32}$$

for $k \geq 1$. Alternatively, using the risk-index (2.6)

$$f(\mathbf{x}, \mathbf{p}) = \frac{LB(X) + E(X) + UB(X)}{3} ,$$

we may compute the *k*-sigma risk indices for both the actions A_1 and A_2, as explained in both the next section and (more elaborately) in Chapter 3, and proceed to make a 'balanced' assessment, Table 2.7 (recommended action in boldface).

	A_1	A_2
0-sigma risk	**488960**	509450
1-sigma risk	**488960**	513927
2-sigma risk	638573	**596553**
3-sigma risk	794873	**679180**

Table 2.7: Balanced Risk Indices
(recommended action bold-faced)

So, if we operationalize risk as probability times consequence [i.e. (2.3)] , we will choose not to invest in protection measures for the type I bridges, that is, we will choose A_1. If we operationalize risk as some worst-case scenario [i.e. the design return period approach (2.4)], we will choose to invest in protection measures for the type I bridges, that is, we will choose A_2. If operationalize risk as a a balanced trade-off between worst-, most likely-, and best-case scenarios [i.e. (2.6)], then the choice of our action will be guided by what we consider to constitute worst- and best-case scenarios. For 1-sigma worst- and best-case scenarios we will have a (slight) preference for action A_1. But as we consider higher order sigma worst- and best-case scenarios, then our preference will shift ever more strongly towards A_2, Table 2.7.

Note that for this particular probabilistic cost-benefit analysis, as we have weighted costs and benefits by way of their plausibility of occurring, the decision theoretical resolution is somewhat low. Stated differently, our decisions are not clear and crisp, as the infrastructural risk-manager is likely to agonize whether or not to invest in additional protection measures for bridges of type I, as the row values in Table 2.7 are relatively close to each other. In Chapter 4 we give another decision theoretical toy-problem, which has a more clear decision theoretical resolution.
In closing, if we only consider the means (2.29), then we are effectively assessing the worst and best case scenarios to be equal to those means (2.6):

$$\frac{LB(X)+E(X)+UB(X)}{3}=\frac{E(X)+E(X)+E(X)}{3}=E(X), \tag{2.33}$$

which is why the 0-sigma row in Table 2.7 gives the mean for both A_1 and A_2, (2.29).

If we consider the plausible best and worst case scenarios to be, respectively, the 1-sigma lower and upper bounds, then for the A_1 column we may compute the best case scenario to be (in terms of total repair costs) to be

$$LB(X)=E(X)-\text{std}(X)=488960-468900=20060, \tag{2.34}$$

and the worst case scenario

$$UB(X)=E(X)+\text{std}(X)=488960+468900=957860. \tag{2.35}$$

Subsitting (2.34) and (2.35) into (2.6), we have that the standard deviation $\mathrm{std}(X)$ in both (2.34) and (2.35) will cancel out:

$$\frac{LB(X)+E(X)+UB(X)}{3}=\frac{[E(X)-std(X)]+E(X)+[E(X)+std(X)]}{3}$$
$$=E(X) \tag{2.36}$$

which is why the 1-sigma row in Table 2.7 gives again the mean of A_1.

However, if we consider the still plausible best and worst case scenarios to be, respectively, the 2-sigma lower and upper bounds, then for the A_1 column we may compute the best case scenario to be (in terms of total repair costs) to be

$$LB(X)=E(X)-2\ \mathrm{std}(X)=488960-937800=-448840, \tag{2.37}$$

which has to be corrected for lower bound undershoot [as negative repair costs equate (unrealistic) profits resulting from having to repair damaged infrastructure]:

$$LB(X)=0, \tag{2.38}$$

and the worst case scenario as

$$UB(X)=E(X)+2\ \mathrm{std}(X)=488960+937800=1426760. \tag{2.39}$$

Subsitting (2.38) and (2.39) into (2.6), we no longer have that the standard deviation $\mathrm{std}(X)$ cancels out:

$$\frac{LB(X)+E(X)+UB(X)}{3}=\frac{0+E(X)+[E(X)+2\ \mathrm{std}(X)]}{3}$$
$$=\frac{2E(X)+2\,\mathrm{std}(X)}{3} \tag{2.40}$$

which is why the 2-sigma row in Table 2.7 gives a value other than the mean of A_1.

3.0 A DECISION THEORETICAL OVERVIEW

In this chapter a theoretical discussion of the Bayesian decision theory is given (van Erp *et al.*, 2016*a*). This is done by relating the Bayesian decision theory to the expected outcome and expected utility theories that came before it. The goal is to develop methods and techniques how to proceed after a stress test would result in a positive or negative result. Questions such as whether safety measures need to be taken to improve the situation (and to which extent) or if actions can be delayed, can be answered by adopting a cost-benefit framework within the Bayesian paradigm.

3.1 Expected Outcome Theory

Expected outcome theory has been around since the 17th century, when the rich merchants of Amsterdam sold and bought expectations as if they were tangible goods. The algorithmic steps of expected outcome theory are very simple:

(1) For each possible decision construct an outcome probability distributions; i.e. for each possible decision, assign to every conceivable contingency both an estimated net-monetary-consequence and a probability.
(2) Choose that decision which maximizes the expectation values (i.e. means) of the outcome probability distributions.

3.2 Bernoulli's Expected Utility Theory

In the 18th century Bernoulli provided a fundamental contribution to expected outcome theory in that he proposed that it were not the actual gains and losses that move us, but rather that it is the utility of these gains and losses that move us. Moreover, Bernoulli offered up a specific function by which to translate these gains and losses to their corresponding utilities:

$$u = q \log \frac{m+x}{m}, \qquad q > 0, \tag{3.1}$$

where q is some scaling constant that falls away in the decision theoretical (in)equalities, m is the initial wealth of the decision maker, and x is either a gain or a loss.

In the utility function (3.1) the initial wealth functions as a reference point in the following sense. For increments x which are small relative to the initial wealth m the utility function (3.1) becomes linear, as losses are weighted the same as corresponding gains. Whereas for increments x which are large relative to the initial wealth m the utility function (3.1) becomes non-linear, as losses are weighted heavier than corresponding gains. So, Bernoulli's utility function predicts that the psychological phenomenon of loss aversion will hold for large consequences like, say, the burning down of a house, but not for small consequences like, say, the breaking of an egg. This is commensurate with our intuition.

Bernoulli, having provided both the concept and the quantification of utilities, proposed his expected utility theory as a straightforward generalization of the expected outcome theory. The algorithmic steps of expected utility theory are as follows (Bernoulli, 1738):

(1) For each possible decision construct an outcome probability distributions; i.e. for each possible decision, assign to every conceivable contingency both an estimated net-monetary-consequence and a probability.
(2) Transform outcome probability distributions to their corresponding utility probability distributions; i.e. convert the outcomes of the outcome probability distributions to their corresponding utilities, using Bernoulli's utility function.
(3) Choose that decision which maximizes the expectation values (i.e. means) of the utility probability distributions.

Bernoulli's utility concept remained uncontested in the centuries that followed his 1738 paper, but the same cannot be said for his utility function (3.1). Even though Bernoulli's utility function has been demonstrated to also hold for sensory stimuli perception (Fancher, 1991), and not only for monetary stimulus perception. Moreover, Bernoulli's utility function may be derived from sound first principles (see Appendix A).

In von Neumann and Morgenstern's reintroduction of Bernoulli's expected utility theory the specific form of the utility function was left unspecified (von Neumann and Morgenstern, 1946). This added degree of freedom in the expected utility theory opened the way for alternative utility functions, like, say, the function (Tversky & Kahneman, 1992):

$$u = (-x)^a, \qquad \text{for } a > 0 \text{ and } x \leq 0. \tag{3.2}$$

This alternative power function (6.2), however, does not, like Bernoulli's utility function (3.1), have an explicit reference point by which to modulate the strength of the loss aversion effect as a function of both the current asset position and the increment in that asset position. Moreover, (3.2) lacks the general validity that (3.1) enjoys as the psycho-physical Fechner-Weber law that guides our human sense perception (Fancher, 1991). Finally, Bernoulli's utility function admits a consistency derivation which shows that the only consistent utility function is either the utility function (3.1) or some transformation thereof (van Erp *et al.*, 2015), and it may be shown that (3.2) does not belong to this class of consistent utility functions.

3.3 Bayesian Decision Theory

The Bayesian decision theory is neo-Bernoullian in that it proposes that the utility function (3.1) is the most appropriate function by which to translate, for a given initial wealth, gains and losses to their corresponding utilities. But it deviates from both the expected outcome and the expected utility theories in that it questions the appropriateness of the criterion of choice where one has to choose that decision that maximizes the expectation values, or, equivalently, the means, of the outcome probability distributions under the different decisions.

3.3.1 The Criterion of Choice as a Degree of Freedom

Let D_1 and D_2 be two actions we have to choose from. Let x_i, for $i = 1, \ldots, n$, and x_j, for $i = 1, \ldots, m$, be the monetary outcomes associated with, respectively, actions D_1 and D_2. Then in the Bayesian decision theory – as in expected outcome and utility theories – one constructs the two outcome distributions that correspond with these decisions:

$$p(x_i \mid D_1), \qquad \text{and} \qquad p(x_j \mid D_2). \tag{3.3}$$

One then proceeds – as in expected utility theory – to map utilities to the monetary outcomes in (3.3), by way of the Bernoulli utility function (3.1). This leaves us with the utility probability distributions:

$$p(u_i \mid A_1), \qquad \text{and} \qquad p(u_j \mid A_2). \tag{3.4}$$

Now, our most primitive intuition regarding the utility probability distributions (3.4) is that the action which corresponds with the utility probability distribution which lies more to the right will also be the action that promises to be the most advantageous. So, when making a decision we ought to compare the positions of the utility probability distributions on the utility axis and then choose that action which maximizes the position of these utility probability distributions.

This all sounds intuitive enough. But how do we define the position of a probability distribution? Ideally we would have some formal (consistency) derivation of what constitutes a position measure of a probability distribution. But in the absence of such a derivation we have to take our recourse to ad hoc common sense considerations. Stated differently, the criterion of choice in our decision theory constitutes a degree of freedom.

3.3.2 The Probabilistic Worst, Most Lkely, and Best Case Scenarios

From the introduction of expected outcome theory in the 17th century and expected utility theory in the 18th century the implicit assumption has been that the expectation value of a given probability distribution is a position of its measure (Jaynes, 2003; Bernoulli, 1738). The expectation value is a measure for the location of the centre of mass of a given probability distribution; as such it may give one a probabilistic indication of the most likely scenario:

$$E(X) = \sum_{i=1}^{n} p_i x_i \ . \tag{3.5}$$

The qualifier 'probabilistic indication' is used here in order to point to the fact that the expectation value, or, equivalently, the mean, need not give a value that one would necessarily expect.
In the Value at Risk (VaR) methodology – used in the financial industry (Davies, 2010) – the probabilistic worst case scenarios are taken as a criterion of choice, rather than the most likely scenarios (in the probabilistic sense). In the VaR methodology the probabilistic worst case scenario is operationalized as the 1% percentile. But instead of percentiles one may also use the confidence lower bound to operationalize a probabilistic worst case scenario.

The absolute worst case scenario is:

$$a = \min(x_1, \ldots, x_n) \tag{3.6}$$

The criterion of choice (3.6) is also known as the minimax criterion of choice (Lindgren, 1993). The *k*-sigma lower bound of a given probability distribution is a given as

$$lb(k) = E(X) - k \ \mathrm{std}(X) \tag{3.7}$$

where k is the sigma level of the lower bound and where, (3.3),

$$\mathrm{std}(X) = \sqrt{\sum_{i=1}^{n} p_i x_i^2 - [E(X)]^2} \tag{3.8}$$

is the standard deviation. The probabilistic worst case scenario then may be quantified as an undershoot corrected lower bound, (3.6) and (3.7):

$$LB(k) = \begin{cases} lb(k), & lb(k) \geq a, \\ a, & lb(k) < a. \end{cases} \tag{3.9}$$

Note that the probabilistic worst case scenario (3.9) holds the minimax criterion of choice (3.6) as a special case for large k in (3.7). For $k = 1$, the criterion of choice (3.9) constitutes a still likely worst case scenario (in the probabilistic sense).

One may also imagine – in principle – a decision problem in which one is only interested in the probabilistic best case scenarios. The absolute best case scenario is:

$$b = \max(x_1, \ldots, x_n) \tag{3.10}$$

The criterion of choice (3.10) is also known as the maximax criterion of choice. The k-sigma upper bound of a given probability distribution is a given as:

$$ub(k) = E(X) + k \ \mathrm{std}(\mathrm{X}) \tag{3.11}$$

where k is the sigma level of the upper bound. The probabilistic best case scenario then may be quantified as an overshoot corrected upper bound, (3.10) and (3.11):

$$UB(k) = \begin{cases} b, & ub(k) > b, \\ ub(\mathrm{k}), & ub(k) \leq b. \end{cases} \tag{3.12}$$

Note that the probabilistic best case scenario (3.12) holds the maximax criterion of choice (3.10) as a special case for large k in (6.11). For $k = 1$, the criterion of choice (3.12) constitutes a still likely best case scenario (in the probabilistic sense).

If one takes as a criterion of choice (3.5), then one neglects what may happen in the worst and the best of worlds. If one takes as a criterion of choice (3.9), then one neglects what may happen in the most likely and the best of worlds. If one takes as a criterion of choice (3.12), then one neglects what may happen in the worst and the most likely of worlds. An exclusive commitment to any of the criteria of choice (3.5), (3.9), (3.12), will necessarily leave out some pertinent information in one's decision theoretical considerations. So, how to untie this Gordian knot?

3.3.2 A Probabilistic Hurwitz Criterion of Choice

In Hurwitz's criterion of choice the absolute worst and best case scenarios are both taken into account; for a balanced pessimism coefficient of $\alpha = 1/2$ we have that, (3.6) and (3.10):

$$\text{Hurwitz's criterion of choice } = \frac{a+b}{2}. \tag{3.13}$$

Now, we may replace the absolute worst and case scenarios in (3.13) with their corresponding probabilistic undershoot and overshoot corrected counterparts, (3.9) and (3.12):

$$\text{probabilistic Hurwitz's criterion of choice } = \frac{LB(k)+UB(k)}{2} \tag{3.14}$$

Under the criterion of choice (3.14) undecidedness between D_1 and D_2 translates to the decision theoretical equality:

$$\frac{LB(k \mid D_1)+UB(k \mid D_1)}{2} = \frac{LB(k \mid D_2)+UB(k \mid D_2)}{2}, \tag{3.15}$$

or, equivalently,

$$LB(k \mid D_1) - LB(k \mid D_2) = UB(k \mid D_2) - UB(k \mid D_1), \tag{3.16}$$

a trade-off between the losses/gains in the probabilistic worst case scenarios (3.9) and the corresponding gains/losses in the probabilistic best case scenarios (3.10). It follows, seeing that (3.13) is a limit case of (3.14), that for a balanced pessimism coefficient of $\alpha = 1/2$ Hurwitz's criterion of choice gives us a balanced trade-off between the differences in the absolute worst case scenarios and the differences in the absolute best case scenarios.

The probabilistic Hurwitz criterion of choice (3.14) translates to, (3.9) and (3.12):

$$\frac{LB(k)+UB(k)}{2} = \begin{cases} E(X), & lb(k) \geq a,\ ub(k) \leq b, \\ \dfrac{a+E(X)+k\ \mathrm{std}(X)}{2}, & lb(k) < a,\ ub(k) \leq b, \\ \dfrac{E(X)-k\ \mathrm{std}(X)+b}{2}, & lb(k) \geq a,\ ub(k) > b, \\ \dfrac{a+b}{2}, & lb(k) < a,\ ub(k) > b. \end{cases} \tag{3.17}$$

It follows that the alternative criterion of choice (3.14), which takes into account what may happen in the worst and the best of worlds, both holds the traditional expected value criterion of choice (3.5) as a special case as well as Hurwitz's criterion of choice with a balanced pessimism factor (3.13). However, it may be found that the criterion of choice (3.14) – and by implication also the Hurwitz criterion of choice (3.13) – is vulnerable to a simple counter-example.

Imagine we have two utility probability distributions having equal lower and upper bounds $LB(k)$ and $UB(k)$, but one distribution being right-skewed and the other being left-skewed. Then the criterion of choice (3.14) will leave one undecided between the two decisions, whereas our intuition would give preference to the decision corresponding with the left-skewed distribution, as the bulk of the probability distribution of the left-skewed distribution will be more to the right than that of the right-skewed distribution.

3.3.3 The Proposed Criterion of Choice

The probabilistic Hurwitz criterion of choice (3.17) is an alternative to the expectation criterion of choice (3.5) which also takes into account the standard deviation of a given probability distributions, by way of the positions of the under and overshoot corrected lower and upper bounds. But the universality of this proposal is compromised by way of the simple counter example of a right-skewed and a left-skewed distribution which have the same lower and upper bounds, (3.9) and (3.12). It follows that a criterion of choice, in order to be universal, should not only take into account the trade-off between the probabilistic worst and best case scenarios, as is done in (3.17), but also the location of the probabilistic bulk of the probability distribution.
The following position measure for a probability distribution accommodates the intuitive preference for the left-skewed distribution of the counter example, while taking into account the probabilistic worst and best cases:

$$\frac{LB(k)+E(X)+UB(k)}{3}=\begin{cases} E(X), & lb(k)\geq a,\ ub(k)\leq b, \\ \dfrac{a+2E(X)+k\ \mathrm{std}(X)}{3}, & lb(k)<a,\ ub(k)\leq b, \\ \dfrac{2E(X)-k\ \mathrm{std}(X)+b}{3}, & lb(k)\geq a,\ ub(k)>b, \\ \dfrac{a+E(X)+b}{3}, & lb(k)<a,\ ub(k)>b. \end{cases} \tag{3.18}$$

Note that the alternative criterion of choice (3.18), which takes into account what may happen in the worst, the most likely, and the best of worlds, both holds the traditional expected value criterion of choice (3.5) as a special case.

In any problem of choice one will endeavour to choose that action which has a corresponding utility probability distribution that is lying most the right on the utility axis; that is, one will choose to maximize their utility probability distributions. In this there is little freedom. But one is free, in principle, to choose the measures of the positions of one's utility probability distributions any way one see fit. Nonetheless, it is held to be self-evident that it is always a good policy to take into account all the pertinent information at hand.

Firstly, if one only maximizes the expectation values of the utility probability distributions, then one will, by definition, neglect the information that the standard deviations of the utility probability distributions have to bear on one's problem of choice.

Secondly, if one only maximizes one of the confidence bounds of the utility probability distributions, while neglecting the other, then one will be performing a probabilistic minimax or maximax analysis, and, consequently, neglect the possibility of either the (perhaps catastrophic) losses in the lower bound or the (pperhaps astronomical) gains in the upper bound.

Thirdly, if one only maximizes the sum of the lower and upper bounds, or a scalar multiple thereof, then one will make a trade-off between the probabilistic worst and best case scenarios. But in the process, one will, for unimodal distributions, be neglecting the location of the bulk of the probability distributions.

In light of the above three considerations the scalar multiple the sum of the undershoot corrected lower bound, expectation value, and overshoot corrected upper bound, that is, (3.18), is currently believed to be the most all-round position measure for a given probability distribution, as it takes into account the position of the probabilistic worst and best case scenarios, (3.9) and (3.12), as well as the position of the probabilistic most likely scenario, (3.5).

3.3.4 The Algorithmic Steps of the Bayesian Decision Theory

The algorithmic steps of the Bayesian decision theory are as follows (van Erp *et al.*, 2015):

(1) For each possible decision construct an outcome probability distribution; i.e. for each possible decision, assign to every conceivable contingency both an estimated net-monetary-consequence and a probability.
(2) Transform outcome probability distributions to their corresponding utility probability distributions; i.e. convert the outcomes of the outcome probability distributions to their corresponding utilities, using Bernoulli's utility function.
(3) Maximize a scalar multiple of the sum of the lower bound, the expectation value, and the upper bound of the utility probability distributions; that is, the criterion of choice (3.18).

Note that the Bayesian decision theory is just Bernoulli's expected utility theory, except for the alternative criterion of choice (3.18) which is to be maximized. But in the case that the *k*-sigma confidence lower bound (3.7) does not undershoot the absolute minimum (3.6) and the confidence upper bound (3.11) does not overshoot the absolute maximum (3.10), then the criterion of choice (3.18) collapses to the expectation value (3.5) and, as a consequence, the Bayesian decision theory becomes equivalent to Bernoulli's expected utility theory.

The Bayesian decision theory is a neo-Bernoullian utility theory which also aims to improve on expected utility theory, as did game theory and prospect theory before it. But in its approach it takes the middle road, just as Daniel Bernoulli himself did when he wrote his St. Petersburg paper, in that it recognizes both the desirability of mathematical first principles as well as the necessity for any mathematical theory of human rationality to be able to stand to the benchmark of our common sense.

3.4 Why the Best-Case Scenario Does Matter

Now, both the risk indices (3.5) and (3.12) may appeal to our intuition. The expectation value (3.5) is a traditional definition of risk, whereas the upper *k*-sigma bound (3.12) – i.e. the 'worst-case scenario' if the x_i pertain to some hazard intensity measure – is a traditional engineering design criterion, as structures are built to withstand return periods of hazard intensity measures and as those return periods typically correspond to some upper *k*-sigma bound, or, equivalently, upper percentile. The expectation value relates to that which is most likely to happen, whereas the upper *k*-sigma bound /percentile informs us about the possible severity of a still plausible worst-case scenario.

However, behavioral economists have shown, by way of hypothetical betting experiments, that expected utility theory, which takes as its implied position measure the expectation value (3.5), may suggest to us decisions which are forcefully rejected by our common sense (Tversky and Kahneman, 1992). The resulting discrepancy between, on the one hand, the predictions made by expected utility theory and, on the other hand, the observed betting preferences in psychological laboratory experiments is the very bedrock upon which the behavioral economy paradigm of the non-rational chooser is founded (Kahneman, 2011).

But one may also interpret this observed discrepancy to be an indication that the expectation value is a suboptimal position measure for at least some probability distributions. This is why we ourselves in the first two years of our decision theoretical research only considered the criterion of choice (3.7) – i.e. the uncorrected 'worst-case scenario' if the x_i pertain to monetary net gains – as we were trying to accommodate these discrepancies between model predictions and empirical data.

After having presented our findings for the first time, however, we came to realize that the probabilistic best-case scenario is also important. If the probabilistic worst-case scenario will compel us to invest in risk mitigation measures, then the probabilistic best-case scenario will caution us to be frugal. If we make a trade-off between this desire to invest in risk mitigation, on the one hand, and frugality, on the other hand, then this trade-off will result in an optimal – because balanced – decision.

For example, say we have decisions D_1 and D_2. Decision D_1 is the decision to keep the status quo; that is, to do nothing. Decision D_2 is the decision to invest an amount of I euros in order to bring down the current probability of some adverse event that will lead to extra repair costs, relative to a base-line repair cost of 0 euros, in (the likely) case this adverse event does not materialize.
Under the criterion of choice (3.12), we will have that we are undecided between D_1 and D_2 if

$$\begin{aligned} E(X \mid D_1) + k\ std(X \mid D_1) &= E(I + X \mid D_2) + k\ std(I + X \mid D_2) \\ &= I + E(X \mid D_2) + k\ std(X \mid D_2), \end{aligned} \tag{3.19}$$

(we can go from the first right-hand to second right-hand because I is a constant), or, equivalently,

$$UB(X \mid D_1) = I + UB(X \mid D_2). \tag{3.20}$$

In words, under the criterion of choice (3.12), we will be inclined to choose D_2 over D_1 if the consequent reduction in risk, that is, the upper bound/upper percentile of the repair cost probability distribution, is smaller than the investment cost I :

$$I < UB(X \mid D_1) - UB(X \mid D_2). \tag{3.21}$$

Now, the repair cost probability distributions under D_1 and D_2 will also admit a lower bound, which because of the right-skewness of the probability distributions under High-Impact Low-Probability (HILP) events will for D_1 undershoot the base-line 0 euro repair costs for $k \geq 1$ sigma bounds

$$LB(X \mid D_1) = E(X \mid D_1) - k \ std(X \mid D_1) < 0 \tag{3.22}$$

and, likewise, for D_2 undershoot the base-line 0 euro repair costs plus I euro investment

$$LB(I + X \mid D_2) = I + LB(X \mid D_2) = I + E(X \mid D_2) - k \ std(X \mid D_2) < I, \tag{3.23}$$

which is why we have to correct for this undershoot and set

$$LB(X \mid D_1) = 0. \tag{3.24}$$

and, likewise, for D_2 undershoot the base-line 0 euro repair costs plus I euro investment

$$LB(I + X \mid D_2) = I + LB(X \mid D_2) = I. \tag{3.25}$$

The undershoot corrected lower bounds (10.17) and (10.18) calls to attention the loss of I euros that the risk reduction investment will incur in the best-case scenario where no adverse event materializes.

This is what we mean if we say that the probabilistic worst-case scenario will compel us to invest in risk mitigation measures, whereas the probabilistic best-case scenario will caution us to be frugal. For as long as inequality (3.21) holds, or, equivalently,

$$\Delta_{UB} = UB(X \mid D_1) - UB(I + X \mid D_2) > 0, \tag{3.26}$$

we will be willing to increase our investments I , but the more we invest, the greater will be the investment 'loss', (3.24) and (3.25),

$$\Delta_{LB} = LB(X \mid D_1) - LB(I + X \mid D_2) = -I \,. \tag{3.27}$$

So if we also take into account the best case scenarios, then the investment attractor (3.26), which follows from the repair cost upper bounds, is counter balanced by the investment loss (3.27), which follows from the repair cost lower bounds.

It may be shown that the trade-off between the investment attractor (3.26) and the investment loss (3.27) may be enforced by maximizing the respective probabilistic Hurwicz criteria:

$$R(D_1) = \frac{LB(X \mid D_1) + UB(X \mid D_1)}{2}, \tag{3.28}$$

and

$$R(D_2) = \frac{LB(I + X \mid D_1) + UB(I + X \mid D_1)}{2}. \tag{3.29}$$

which give us an investment willingness which is half the investment willingness under (3.21)

$$I < \frac{UB(X \mid D_1) - UB(X \mid D_2)}{2}. \tag{3.30}$$

However, it may be shown, as discussed in Chapter 6, that the probabilistic Hurwicz criterion is vulnerable to a simple counter example, which is why we recommend the alternative risk index (3.18):

$$R(D_i) = \frac{LB(X \mid D_i) + E(X \mid D_i) + UB(X \mid D_i)}{3}. \tag{3.31}$$

This risk index not only makes the trade-off between the investment attractor (3.26) and the investment loss (3.27), but also takes into account that which is most likely to happen, by way of the additional inclusion of the expectation value (3.5).

3.5 Discussion

Although it seems hardly conceivable today, nonetheless, the historical record shows clearly and repeatedly that the notion of ‘expectation of profit’ to these first workers in probability theory was maybe even more intuitive than the notion of probability itself (Jaynes, 2003). Moreover, it was so obvious to many that a person acting in pure self-interest should always behave in such a way as to maximize the expected profit that the prosperous merchants in 17th century Amsterdam bought and sold mathematical expectations as if they were tangible goods.

A new insight into risk science came in 1738 with the St Petersburg paper (Bernoulli, 1738). This paper is truly important on the subject of risk as well as on human behavior, since it introduces the pivotal idea that the true value to a person, of receiving a certain amount of money, is not measured simply by the amount received; it also depends upon how much he already has: “*Utility resulting* *from any small increase in wealth will be inversely proportionate to the quantity of goods previously possessed.*"

This idea is as simple as it is intuitive, for it stands to reason that a poor person will assign a much higher utility to the monetary gain of a thousand euros than a billionaire will. A modern economist is expressing the same idea when he speaks of the 'diminishing marginal utility of wealth'. So where decision theory, by way of the maximization of the expectation of profit, hitherto only had taken profits and the probabilities of these profits materializing into account, there Daniel Bernoulli identified the initial wealth of the decision maker as a third decision theoretical factor of consequence.

Bernoulli's expected utility theory laid the intellectual groundwork for many aspects of micro-economic theory and decision-theory. Furthermore, it also, by way of its utility function – that function which measures that which cannot be counted – also managed to elevate psychology in 1860 from a mere metaphysical pastime to an exact science (Fancher, 1990), as it was found by Fechner that Bernoulli's utility function may model our perception of sensory stimuli for a given background stimulus intensity (Masin et al., 2009); e.g. the decibel scale is an instance where Bernoulli's utility function converts objective loudness stimuli to a corresponding subjective scale.
Moreover, with the introduction of operational risk management in the mid-20th century, in the aftermath of the second world war and the beginning of the atomic era, Bernoulli's expected utility theory has remained as relevant as it ever was. For many improvements by many authors have been proposed to Bernoulli's initial 1738 proposal; the most notable of these proposed improvements being game theory and their off-shoots (von Neumann & Morgenstern, 1946; Savage, 1954) and prospect theory (Tversky and Kahneman, 1992).

Von Neumann and Morgenstern formulated a lofty predictive mathematical theory on their axiomatic scaffolding and proclaimed that their theory had superseded Bernoulli's expected utility theory, as a more general theory (von Neumann & Morgenstern, 1946). But Kahneman and Tversky then were quick to point out that both Von Neumann and Morgenstern's game theory and Bernoulli's expected utility theory could be demonstrated to violate common sense rationality in certain instances and, as a reaction, counter postulated that human rationality can never be captured by simple mathematical maximization principles (Bernstein, 1998).

Kahneman and Tversky then proceeded to propose a mathematical descriptive decision theory that was rooted in empirical observation of psychological betting experiments, rather than abstract mathematical first principles (Tversky and Kahneman, 1992). So persuasive was the case made by Kahneman and Tversky that the descriptive paradigm of behavioral economics came to dominate the decision theoretical field.

But it has been found that cumulative prospect theory, which is not build from first principles, but, rather, is built from the outset to accommodate the Ellsberg and Allais paradox, as well as the specific convex-down and concave-up shape of the fair probability curves in certainty bets, may be replaced by Bernoulli's original proposal[3] (Bernoulli, 1738), with the adjusted criterion of choice that the confidence bound overshoot corrected position measure of the utility probability distribution (3.18), should be maximized; rather than the expected utility value (3.5). This, mathematically trivial adjustment of Bernoulli's expected utility theory[4] – that is, the Bayesian decision theory – accommodates the experimental results which were in contradiction with von Neumann and Morgenstern's expected utility proposal and, moreover, is built from first principles (van Erp *et al.*, 2014).

So just as prospect theory, in its initial inception, was mainly a reaction to von Morgenstern and Neumann's game theory, which put too high a premium on mathematics, while failing to appeal to our common sense. So the Bayesian decision theory (van Erp et al., 2015) is a reaction to Kahneman and Tversky's prospect theory, which puts too high a premium on bare bone empiricism of hypothetical betting experiments, while failing to appeal to our mathematical need for compelling first principles.

[3] Bernoulli's original 1738 article was translated, from its original Latin in English, only as late as 1954.
[4] Note that the Ellsberg and some of the Allais paradoxes were already accommodated by Bernoulli's original proposal, whereas it is the adjusted criterion of choice, where (3.18) is maximized rather than (3.5), which allows us to reproduce, from first principles, the specific convex-down and concave-up shape of the fair probability curves, as found in the experimental certainty bets, as well as the remaining Allais paradoxes not accounted for by Bernoulli's expected utility theory.

4.0 MODELLING INVESTMENT WILLINGNESS

In this chapter a simple toy problem is discussed by way of the three decision theories: expected outcome theory, expected utility theory, and Bayesian decision theory (van Erp *et al.*, 2016*b*). In the first part of this chapter the analytical solution of the investment willingness, or, equivalently, hypothetical benefits, is given. In the second part of this chapter some numerical values will be inserted into these analytical expressions and the resulting numerical solutions will then be discussed.

4.1 The Problem of Choice

The Bayesian framework is now applied to a problem of choice in which a decision maker must decide on how much he is willing to invest in order to reduce the probability of a type II risk event (High Impact Low Probability HILP event) occurring. The two decisions under consideration in this simple scenario are:

$D_1 =$ keep the status quo,
$D_2 =$ improve barrier for type II event.

The possible outcomes in the risk scenario remain the same under either decision, and therefore are not dependent upon the particular decision taken. These outcomes are

$O_1 =$ catastrophic type II event occurs,
$O_2 =$ no type II event.

The hypothetical damages associated with these outcomes are,

$$O_1 = -x \text{ euros},$$
$$O_2 = 0 \text{ euros}, \tag{4.1}$$

respectively, and the investment costs associated with the additional improvement of the type II event barriers are expressed by the parameter

$$I = \text{ investment costs}. \tag{4.2}$$

The decision whether to improve the type II event barriers or not is of influence on the probabilities of the respective outcomes. Under the decision to make no additional investments in the type II event barriers and keep the status quo, D_1, the probabilities of the outcomes will be, say,

$$P(O_1 \mid D_1) = \theta,$$
$$P(O_2 \mid D_1) = 1 - \theta. \tag{4.3}$$

Under the decision to improve the type II event barriers, D_2, the probability of the catastrophic type II event will be decreased, say,

$$P(O_1 \mid D_2) = \phi\,,$$
$$P(O_2 \mid D_2) = 1 - \phi\,. \tag{4.4}$$

where $\phi < \theta$. Stated differently, the proposed barrier improvements will decrease the chances of the catastrophic type II event by a factor of $c = \theta/\phi$.

In what follows, the solution of this problem of choice will be given for expected outcome theory, expected utility theory, and Bayesian decision theory. These solutions will be given in terms of variable x, θ, and ϕ, respectively, (4.1), (4.3), and (4.4).

4.2 The Expected Outcome Theory solution

The prosperous merchants in the 17th century Amsterdam bought and sold expectations as if they were tangible goods. It seemed obvious to many that a person acting in pure self-interest should always behave so as to maximize his expected profit (Jaynes, 2003).

Combining (4.1) through (4.4), one may construct the outcome probability distributions under the decisions D_1 and D_2:

$$p(O_i \mid D_1) = \begin{cases} \theta, & O_1 = -x, \\ 1-\theta, & O_2 = 0, \end{cases} \tag{4.5}$$

and

$$p(O_j \mid I, D_2) = \begin{cases} \phi, & O_1 = -x - I, \\ 1-\phi, & O_2 = -I, \end{cases} \tag{4.6}$$

where in (4.6) one may explicitly conditionalize on the investment parameter I, which is to be to estimated. The expected outcomes of these probability distributions are, respectively (Lindgren, 1993):

$$E(O \mid D_1) = -\theta\, x\,, \tag{4.7}$$

And

$$E(O \mid I, D_2) = -\phi\, x - I\,. \tag{4.8}$$

The decision theoretical equality

$$E(O \mid D_1) = E(O \mid I, D_2) \tag{4.9}$$

represents the equilibrium situation, where it will be undecided to choose between the decision to keep the status quo D_1 and the decision to invest in additional barrier improvements D_2. Now, if one solves for I in (4.9), by way of (4.7) and (4.8):

$$I = (\theta - \phi)x , \tag{4.10}$$

then we find that investment where one will be undecided between either decision.

Stated differently, any investment cost smaller than (4.10) will turn (4.9) into an inequality, where D_2 becomes more attractive than D_1. It follows that the equilibrium investment (4.10) is also the maximal investment one will be willing to make in order to improve the type II event barriers, or, equivalently, (4.10) is the hypothetical benefit of the type II event barrier improvement.

4.3 The Expected Utility Theory Solution

For a rich man hundred euros is an insignificant amount of money. So, the prospect of gaining or losing a hundred euros will fail to move the rich man, that is, an increment of hundred euros for him has a utility which tends to zero. For the poor man a hundred euros will be a significant amount of money. So, the prospect of gaining or losing hundred euros will most likely move the poor man to action. It follows that for him an increment of a hundred euros has a utility significantly greater than zero.

In 1738 Daniel Bernoulli derived the utility function for the subjective value of objective moneys by way of a variance argument, in which he considered the subjective effect of a given fixed monetary increment c for two persons holding different initial wealths. Based on this variance argument he derived the utility function of going from an initial asset position x to the asset position $x + c$:

$$u(x, x + c) = q \log \frac{x + c}{x} \tag{4.11}$$

where q is some scaling constant greater than zero (Bernoulli, 1738).

In expected utility theory the expected values of the utility probability distributions are maximized. Assuming that the decision maker has a total wealth, that is, an actual income and asset portfolio, of

$$M = m \text{ euros}, \tag{4.12}$$

then, using (4.11), or, equivalently,

$$U_i = q \log \frac{M + O_i}{M} , \tag{4.13}$$

one may construct from (4.5) and (4.6) the corresponding utility probability distributions as:

$$p(U_i \mid D_1) = \begin{cases} \theta, & U_1 = q \log \dfrac{m - x}{m}, \\ 1 - \theta, & U_2 = 0, \end{cases} \tag{4.14}$$

And

$$p\left(U_j \mid I, D_2\right)=\begin{cases}\phi, & U_1 = q\log\dfrac{m-x-I}{m},\\ 1-\phi, & U_2 = q\log\dfrac{m-I}{m}.\end{cases} \tag{4.15}$$

The expected outcomes of the utility probability distributions are, respectively (Lindgren, 1993):

$$E\left(U \mid D_1\right)=q\left(\theta \log\frac{m-x}{m}\right) \tag{4.16}$$

and

$$E\left(U \mid I, D_2\right)=q\left(\phi \log\frac{m-x-I}{m-I}+\log\frac{m-I}{m}\right). \tag{4.17}$$

The decision theoretical equality

$$E\left(U \mid D_1\right)=E\left(U \mid I, D_2\right) \tag{4.18}$$

represents the equilibrium situation, between the decision to keep the status quo D_1 and the decision to invest in additional barriers D_2. Now, if one substitutes (4.16) and (4.17) into (4.18), then one obtains the closed expression for that investment value where one is indifferent between either decision:

$$\log\frac{m-I}{m}=\theta \log\frac{m-x}{m}-\phi \log\frac{m-x-I}{m-I}. \tag{4.19}$$

Any investment cost smaller than the numerical solution of I in (4.19) will turn (4.18) into an inequality, where D_2 becomes more attractive than D_1. It follows that the equilibrium investment (4.19) is also the maximal investment one will be willing to make to improve the type II event barriers, or, equivalently, (4.19) is the hypothetical benefit of the type II event barrier improvement.

4.4 The Bayesian Decision Theory Solution

In Bayesian decision theory the scaled sum of the confidence bounds and the expectation value of the utility probability distributions is maximized as the risk measure that captures the position of the underlying utility probability distribution (see section 3.3.3):

$$R\left(U \mid k, D_j\right)=\frac{LB\left(k \mid D_j\right)+E\left(U \mid D_j\right)+UB\left(k \mid D_j\right)}{3}, \tag{4.20}$$

where the lower confidence bound is corrected for undershooting the worst possible outcome a, (3.6):

$$LB(U \mid D_j) = \begin{cases} a, & E(U \mid D_j) - k \ \text{std}(U \mid D_j) < a, \\ E(U \mid D_j) - k \ \text{std}(U \mid D_j), & E(U \mid D_j) - k \ \text{std}(U \mid D_j) \geq a, \end{cases} \tag{4.21}$$

and the upper confidence bound is corrected for overshooting the best possible outcome b, (3.10):

$$UB(U \mid D_j) = \begin{cases} E(U \mid D_j) + k \ \text{std}(U \mid D_j), & E(U \mid D_j) + k \ \text{std}(U \mid D_j) \leq b, \\ b, & E(U \mid D_j) - k \ \text{std}(U \mid D_j) > b. \end{cases} \tag{7.22}$$

Substituting (4.21) and (4.22) into (4.20), one obtains the risk index:

$$R(U \mid k, D_j) = \begin{cases} E(U \mid D_j), & \text{Neither overshoot nor undershoot,} \\ \dfrac{a + 2E(U \mid D_j) + k \ \text{std}(U \mid D_j)}{3}, & \text{Undershoot and no overshoot,} \\ \dfrac{2E(U \mid D_j) - k \ \text{std}(U \mid D_j) + b}{3}, & \text{Overshoot and no undershoot,} \\ \dfrac{a + E(U \mid D_j) + b}{3}, & \text{Both overshoot and undershoot.} \end{cases} \tag{7.23}$$

where it is noted that the first row of (4.23) corresponds with the expected utility theory criterion of choice (Jaynes, 2003), and the fourth row is a kind of adjusted Hurwitz criterion of choice, which may differentiate two probability distributions which have the same minimal and maximal values while at the same time having an opposite skewness.

In the toy-problem under consideration a simple type II risk scenario is modelled, which is typically a high impact low probability scenario; that is, both large monetary costs and small probabilities for the high-impact event, or, equivalently, on the impact side (7.1), $x >> 0$ and, on the probability side (4.3) and (4.4), $\theta, \phi << 0.5$. Stated differently, the utility probability distributions (4.14) and (4.15) under consideration will both be highly skewed to the left and, as a consequence, will lead to the third condition in (4.23):

$$R(U \mid D_j) = \frac{2E(U \mid D_j) - k \ \text{std}(U \mid D_j) + b}{3}. \tag{4.24}$$

The best possible outcome under decision D_1 is (4.14):

$$b = 0, \tag{4.25}$$

and the standard deviation of (4.14) is (Lindgren, 1993):

$$\mathrm{std}(U \mid D_1) = -q\sqrt{\theta(1-\theta)}\log\frac{m-x}{m}\,. \tag{4.26}$$

So, the risk index under the decision to keep the status quo is, substituting (4.16), (4.25), and (4.26), into (4.24):

$$R(U \mid D_1) = \frac{q}{3}\left[2\theta + k\,\sqrt{\theta(1-\theta)}\right]\log\frac{m-x}{m} \tag{4.27}$$

The best possible outcome under decision D_2 is (4.15):

$$b = q\log\frac{m-I}{m}\,, \tag{4.28}$$

and the standard deviation of (4.15) is (Lindgren, 1993):

$$\mathrm{std}(U \mid I, D_2) = -q\sqrt{\phi(1-\phi)}\,\log\frac{m-x-I}{m-I}\,. \tag{4.29}$$

So, the risk index under the decision invest in additional barriers is, substituting (4.17), (4.28), and (4.29), into (4.24):

$$R(U \mid I, D_2) = \frac{q}{3}\left[2\phi + k\,\sqrt{\phi(1-\phi)}\right]\log\frac{m-x-I}{m-I} + q\log\frac{m-I}{m}\,. \tag{4.30}$$

The decision theoretical equality

$$R(U \mid D_1) = R(U \mid I, D_2) \tag{4.31}$$

represents the equilibrium situation, between the decision to keep the status quo D_1 and the decision to invest in additional risk barriers D_2. Now, if one substitutes (4.27) and (4.30) into (4.31), then one obtains the closed expression for that investment value which will leave one undecided:

$$\log\frac{m-I}{m} = \frac{1}{3}\left\{\left[2\theta + k\,\sqrt{\theta(1-\theta)}\right]\log\frac{m-x}{m} - \left[2\phi + k\,\sqrt{\phi(1-\phi)}\right]\log\frac{m-x-I}{m-I}\right\}. \tag{4.32}$$

Any investment smaller than the numerical solution of I in (4.32) will turn (4.31) into an inequality, where D_2 becomes more attractive than D_1. It follows that the equilibrium investment (4.32) is also the maximal investment one will be willing to make to improve the type II event barriers, or, equivalently, (4.32) is the hypothetical benefit of the type II event barrier improvement.

Note that the 'Weber-constant' q has fallen away in both the decision theoretical equalities (4.19) and (4.32). This will hold in general, as both the expectation values and standard deviations in (4.14) and (4.25) are both linear in the unknown constant q. It follows that one may set, without any loss of generality, q equal to one.

4.5 Some Numerical Results

In the simple toy-problem one has a decision maker who must decide on how much he is willing to invest in further improvements of his type II risk barriers.

4.5.1 Removing Unsafety

After the great Dutch flooding in 1953 the 'Oosterschelde Waterkering' was built. This was a movable dike that allowed for an improved safety from $1/100$ to $1/4000$, while keeping the Oosterschelde connected to the North Sea. This open connection to the North Sea was decided upon in order to keep the salt-sea ecological system of the Oosterschelde river intact.

The total costs of the Oosterschelde Waterkering where about 2.5 billion euros. The bulk of these costs where due to the movable character of this dike. Had the Dutch government decided to build an unmovable dike, then the costs would only have been about 175 million euros.

The total value of the assets at risk at the time where about $1/20$th of the GDP at that time,

$$x = 3.75 \times 10^{9} \text{ euros.} \tag{4.33}$$

The wealth of the decision maker, that is, the Dutch government, was about 40% of the Dutch GDP at that time, aggregated over a period of five years; five years being the total construction time of the movable Oosterschelde dyke:

$$m = 1.5 \times 10^{11} \text{ euros.} \tag{4.34}$$

The status quo probability of a catastrophic flooding had right after the great flood been estimated to be, (4.3):

$$\theta = \frac{1}{100}, \tag{4.35}$$

whereas the probability of the catastrophic flooding under the improved flood defences had been estimated as, (4.4):

$$\phi = \frac{1}{4000}. \tag{4.36}$$

Substituting the values (4.33) through (4.36) into (4.10), (4.19), and (4.32), one obtains the following solutions for the maximal investment willingness, or, equivalently, the hypothetical benefit, I:

- Expected outcome theory:
    - Any sigma level: $I = 36.6 \times 10^6$ euros

and

- Expected utility theory:
    - Any sigma level: $I = 37.0 \times 10^6$ euros

and

- Bayesian decision theory:
    - 1-sigma level: $I = 129.8 \times 10^6$ euros
    - 2-sigma level: $I = 234.9 \times 10^6$ euros
    - 3-sigma level: $I = 340.1 \times 10^6$ euros

It is noted here that after the great Dutch flood the discussion was not whether to build additional flood defences or not, but, rather, whether or not to choose for the expensive solution over the `cheap' solution, which would keep the Oosterschelde salt-sea ecosystem intact. Under expected utility theory the cheap solution of an unmovable dyke would have been too expensive by a factor of three, whereas under Bayesian decision theory the cheap solution was well within the 2-sigma bounds.

4.5.2 Maintaining Safety

The current total value of the assets at risk in the Oosterschelde region are about $1/20$th of the current GDP, (39):

$$x = 30 \times 10^9 \text{ euros.} \tag{4.37}$$

The wealth of the decision maker, that is, the Dutch government, is about 20% of the current Dutch GDP:

$$m = 1.2 \times 10^{11} \text{ euros.} \tag{4.38}$$

If one assumes the current probability of a catastrophic flooding to be $1/4000$, and if one assumes that in the absence of any maintenance the flood defences will have deteriorated such that the probability of a catastrophic flooding will have doubled to $1/2000$ five years from now. Then $\sqrt[5]{2}$ is the implied ‘doubling’ one year away from the latest maintenance round. Using this doubling factor of $\sqrt[5]{2}$, the probability of a catastrophic flooding becomes, (4.3):

$$\theta = \frac{\sqrt[5]{2}}{4000} . \quad (4.39)$$

If one assumes that the probability of a catastrophic flooding under the flood defence maintenance is our current probability of a catastrophic flooding, (4.4):

$$\phi = \frac{1}{4000} . \quad (4.40)$$

Then one has a scenario in which one wishes to prevent a current situation, which is very safe (4.40), from sliding into a somewhat less safe situation (4.39).

Substituting the values (4.37) through (4.40) into (4.10), (4.19), and (4.32), one obtains the following solutions for the maximal investment willingness, or, equivalently, the hypothetical benefit, I :

- Expected outcome theory:
  - Any sigma level: $I = 1.1 \times 10^6$ euros

and

- Expected utility theory:
  - Any sigma level: $I = 1.3 \times 10^6$ euros

and

- Bayesian decision theory:
  - 1-sigma level: $I = 13.9 \times 10^6$ euros
  - 2-sigma level: $I = 26.9 \times 10^6$ euros
  - 3-sigma level: $I = 39.8 \times 10^6$ euros

It is noted here that in order to obtain the very real safety benefit of preventing the probability of a catastrophic flooding of $\phi = 1/4000$ from sliding to $\theta = \sqrt[5]{2}/4000$, expected utility theory is not willing to invest more than 1.3 million euros, whereas Bayesian decision theory with utility transformation, under a 2-sigma safety level, is willing to invest 26.9 million euros for the safety maintenance of the Oosterschelde Waterkering.

So it would seem that Bayesian decision theory solution is more commensurate with observed safety management practices, seeing that the Dutch government yearly spends about 20 million euros to keep the Oosterschelde Waterkering maintained.

4.6 Modelling the Disproportionality Factor

In this chapter we have compared expected outcome theory, expected utility theory, and Bayesian decision theory, for a simple toy-problem in which we look at the investment willingness to avert a high impact low probability event. We have demonstrated that the adjusted criterion of choice, in which scalar multiples of the sum of the lower confidence bound, expectation value, and upper confidence bound of the utility probability distributions are maximized, though mathematical trivial (van Erp et al., 2015), has non-trivial practical implications for the modelled investment willingness, or, equivalently, for the modelled hypothetical benefits.

In closing, based on the numerical results in the previous section it may be argued that it is the insufficiency of the expectation value (3.5) as an index of risk that forces a cost benefit analysis to introduce disproportionality factors as an ad hoc fix-up (Thomas & Jones, 2010); that is, the disproportionality factors are needed in cost benefit analyses because the currently computed hypothetical benefits (4.20) may severely underestimate the actual hypothetical benefits (4.32) which are computed by way of the more realistic index of risk (4.23).

In the case where one restricts oneself to outcome probability distributions, then the hypothetical benefits in a cost benefit analysis are computed as the difference between the expectation value of the outcome under the additional safety barriers and the expectation value of the outcome under the current status quo, (4.10):

$$I = (\theta - \phi)x\,. \tag{4.41}$$

But under the alternative index of risk (4.23), which not only takes into account the most likely trajectory but also the worst and best case scenarios, the corresponding hypothetical benefits may be computed as [compare with (4.32)]:

$$I = \frac{1}{3}\left[\left(2\theta + k\sqrt{\theta(1-\theta)}\right) - \left(2\phi + k\sqrt{\phi(1-\phi)}\right)\right]x\,. \tag{4.42}$$

So, for the outcome probability distributions (4.5) and (4.6) the alternative criterion of choice (4.23) implies a theoretical disproportionality factor *DF* which is the ratio of (4.42) to (4.41):

$$DF = \frac{1}{3}\left[2 + k\frac{\sqrt{\theta(1-\theta)} - \sqrt{\phi(1-\phi)}}{(\theta - \phi)}\right], \tag{4.43}$$

In (Goose, 2006; Rushton, 2006) appropriate disproportionality factors are recommended based on common sense considerations. Alternatively, in (4.43) we have an instance where the appropriate disproportionality factor may be derived for a specific risk scenario, by comparing the hypothetical benefits under the traditional criterion of choice (3.5) and the alternative criterion of choice (4.23).

5.0 HIGH IMPACT LOW PROBABILITY EVENTS VS. LOW IMPACT HIGH PROBABILITY EVENTS

Due to many consistency problems with, on the one hand, predictive risk analysis and expected utility theory and, on the other hand, common sense considerations of risk acceptability in some well-constructed counter examples, it has been difficult to maintain the case for the feasibility of a predictive theory of choice. Even to such an extent that one may read in a text book like (Reith, 2009) that the utility of the notion of 'risk' lies not in its ability to correctly predict future outcomes, but rather in its ability to provide a basis for decision-making.

In Van Erp *et al.*(2015) it is postulated that it may very well be the insufficiency of the expectation value (3.5) as an index of risk:

$$E(X)=\sum_{i=1}^{n} p_i x_i \ , \tag{5.1}$$

which lies at the heart of the inability of expected utility theory to model are choice preferences. For it is found that the alternative risk index (3.18):

$$\frac{LB(k)+E(X)+UB(k)}{3}=\begin{cases} E(X), & lb(k)\geq a,\ \ ub(k)\leq b, \\ \dfrac{a+2E(X)+k\ \text{std}(X)}{3}, & lb(k)< a,\ \ ub(k)\leq b, \\ \dfrac{2E(X)-k\ \text{std}(X)+b}{3}, & lb(k)\geq a,\ \ ub(k)> b, \\ \dfrac{a+E(X)+b}{3}, & lb(k)< a,\ \ ub(k)> b. \end{cases} \tag{5.2}$$

accommodates the prospect theoretical counter examples to expected utility theory; i.e. the reflection effect and the inverted *S*-shape in certainty bets (van Erp *et al.*, 2015). Moreover, the alternative risk index may also shed some light on some common sense observations like the fact that most people judge a HILP (high impact low probability) event as more undesirable than a LIHP (low impact high probability) event, even if the expected consequence of the two events would be exactly the same.

In this section it will be demonstrated that the alternative risk index (5.2) will assign higher risks to HILP (high impact low probability) events than to LIHP (low impact high probability) events, even if the expectation values of both events are so constructed that they are equal for both the HILP and the LIHP event.

5.1 Outcome Probability Distributions with Equal Expectation Values

Say, one has two dichotomous events, of which one is a LIHP, or, equivalently, type I event T_1, and the other is a HILP, or, equivalently, type II event T_2. If (1) the negative consequence of the HILP event occurring is n times the negative consequence of the LIHP event occurring, (2) the consequence of the not-occurring of either event is zero, and (3) the probability of the HILP event is an nth fraction of the probability of the LIHP event, that is,

$$p(O_i \mid T_1) = \begin{cases} \theta, & O_1 = x, \\ 1-\theta, & O_2 = 0, \end{cases} \tag{5.3}$$

$$p(O_i \mid T_2) = \begin{cases} \dfrac{\theta}{n}, & O_1 = n\,x, \\ 1-\dfrac{\theta}{n}, & O_2 = 0. \end{cases} \tag{5.4}$$

Then both outcome probability distributions will admit the same expectation values, that is,

$$E(O \mid T_1) = x\theta = nx\frac{\theta}{n} = E(O \mid T_2)\,. \tag{5.5}$$

Note that in the outcome probability distributions (5.3) and (5.4) the negative consequences are represented by positive numbers, rather than negative numbers. As a consequence the risk index (60) is to be minimized, rather than maximized.

5.2 The Risk of the LIHP Event

Say that the LIHP event has a probability of occurring of

$$\theta = \frac{1}{2}, \tag{5.6}$$

Then the expectation value and standard deviation of the corresponding outcome probability distribution (5.5) is given as:

$$E(O \mid T_1) = \frac{x}{2} \tag{5.7}$$

and

$$\text{std}(O \mid T_1) = x\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} = \frac{x}{2}\,. \tag{5.8}$$

So the k-sigma confidence bounds of (5.3) under (5.6) are given as (5.7) and (5.8):

$$lb(k \mid T_1) = E(O \mid T_1) - k\,\text{std}(O \mid T_1) = \frac{1-k}{2}x \tag{5.9}$$

and

$$ub(k \mid T_1) = E(O \mid T_1) + k\,\mathrm{std}(O \mid T_1) = \frac{1+k}{2}x. \tag{5.10}$$

If one corrects for lower bound undershoot and upper bound overshoot, then the corrected confidence bounds are (5.3) and (5.9):

$$\begin{aligned} LB(k \mid T_1) &= \begin{cases} 0, & lb(k \mid T_1) < 0, \\ lb(k \mid T_1), & lb(k \mid T_1) \geq 0, \end{cases} \\ &= \begin{cases} 0, & k \geq 1, \\ \dfrac{1-k}{2}x, & k < 1, \end{cases} \end{aligned} \tag{5.11}$$

and (5.4) and (5.10):

$$\begin{aligned} UB(k \mid T_1) &= \begin{cases} ub(k \mid T_1), & ub(k \mid T_1) \leq x, \\ x, & ub(k \mid T_1) > x, \end{cases} \\ &= \begin{cases} \dfrac{1+k}{2}x, & k < 1, \\ x, & k \geq 1. \end{cases} \end{aligned} \tag{5.12}$$

Substituting (5.7), (5.11), and (5.12) into the left-hand side of (5.2) one obtains a computed risk for the LIHP event of:

$$R(O \mid k, T_1) = \frac{LB(k \mid T_1) + E(O \mid T_1) + UB(k \mid T_1)}{3} = \frac{x}{2}, \qquad \text{for any sigma level } k. \tag{5.13}$$

So, for the particular case of (5.3) and (5.6) the alternative criterion of choice (5.2) collapses to the traditional expectation value (5.7) as the criterion of choice.

5.3 The Risk of the HILP Event

The HILP event has a n time smaller probability of occurring than the LIHP event, and the consequence of the HILP event is n time larger than the consequence of the LIHP event, (8.3) and (8.4). The expectation value and standard deviation of the HILP outcome probability distribution is given as (8.5):

$$E(O \mid T_2) = \frac{x}{2} \tag{5.14}$$

and

$$\mathrm{std}(O \mid T_2) = nx\sqrt{\frac{1}{2n}\left(1-\frac{1}{2n}\right)} = \frac{x\sqrt{2n-1}}{2}. \tag{5.15}$$

So the k-sigma confidence bounds of (5.4) under (5.6) are given as (5.14) and (5.15):

$$lb(k \mid T_2) = E(O \mid T_2) - k\,\mathrm{std}(O \mid T_2) = \frac{1-k\sqrt{2n-1}}{2}x \tag{5.16}$$

and

$$ub(k \mid T_2) = E(O \mid T_2) + k\,\mathrm{std}(O \mid T_2) = \frac{1+k\sqrt{2n-1}}{2}x. \tag{5.17}$$

If one corrects for lower bound undershoot and upper bound overshoot, then the corrected confidence bounds are (5.4) and (5.16):

$$\begin{aligned} LB(k \mid T_2) &= \begin{cases} 0, & lb(k \mid T_2) < 0, \\ lb(k \mid T_2), & lb(k \mid T_2) \geq 0, \end{cases} \\ &= \begin{cases} 0, & k \geq \dfrac{1}{\sqrt{2n-1}}, \\ \dfrac{1-k\sqrt{2n-1}}{2}x, & k < \dfrac{1}{\sqrt{2n-1}}, \end{cases} \end{aligned} \tag{5.18}$$

and (5.4) and (5.17):

$$\begin{aligned} UB(k \mid T_2) &= \begin{cases} ub(k \mid T_2), & ub(k \mid T_2) \leq nx, \\ nx, & ub(k \mid T_2) > nx, \end{cases} \\ &= \begin{cases} \dfrac{1+k\sqrt{2n-1}}{2}x, & k < \sqrt{2n-1}, \\ nx, & k \geq \sqrt{2n-1}. \end{cases} \end{aligned} \tag{5.19}$$

Substituting (5.14), (5.18), and (5.19) into the left-hand side of (5.2) one may obtain for the HILP event the following risk index as function of the chosen k-sigma level:

$$\frac{LB(k \mid T_2) + E(O \mid T_2) + UB(k \mid T_2)}{3} = \begin{cases} \frac{x}{2}, & k < \frac{1}{\sqrt{2n-1}}, \\ \frac{x}{6}\left(2 + k\sqrt{2n-1}\right), & \frac{1}{\sqrt{2n-1}} \leq k < \sqrt{2n-1}, \\ \frac{x}{6}(1 + 2n), & k \geq \sqrt{2n-1}. \end{cases}$$

(5.20)

For both large n, that is, pronounced HILP events, and k-sigma levels in the reasonable range of, say, 1 to 6, or, equivalently, $1/\sqrt{2n-1} \leq k < \sqrt{2n-1}$, the risk (5.2) of the HILP event translates to

$$R(O \mid k, T_2) = \frac{LB(k \mid T_2) + E(O \mid T_2) + UB(k \mid T_2)}{3} = \frac{x}{6}\left(2 + k\sqrt{2n-1}\right). \tag{5.21}$$

So, for the particular case of (5.4), (5.6), and a variable n much greater than one, i.e. $n >> 1$, the alternative criterion of choice (5.2) goes to a value which differs markedly from the traditional expectation value (5.7).

5.4 Comparing the Risks of the LIHP and HILP Events

The LIHP (low impact high probability) event T_1 is operationalized in this section as an event having a probability of $1/2$ and a loss of x, (5.3) and (5.6). The HILP (high impact low probability) event T_2 is operationalized relative to the LIHP event as the event which has a probability which is n times smaller and a loss which is n times x greater, (5.4) and (5.6); i.e. the more severe the HILP event the less likely it occurrence.

The expectation values of the LIHP and HILP events are equal, (5.5). Nonetheless, for a given 'severity' variable n greater than one, the alternative risk index (5.2) will assign a higher risk to the HILP event than to the LIHP event by a factor of, (5.13) and (5.21):

$$\frac{\text{Risk}_{HILP}}{\text{Risk}_{LIHP}} = \frac{R(O \mid k, T_2)}{R(O \mid k, T_1)} = \frac{1}{3}\left(2 + k\sqrt{2n-1}\right) \tag{5.22}$$

If we plot for $k = 1,\ 2,\ 3,\ 6$ sigma levels the risk ratio (5.22) as a function of the severity variable n, we obtain Figure 5.1:

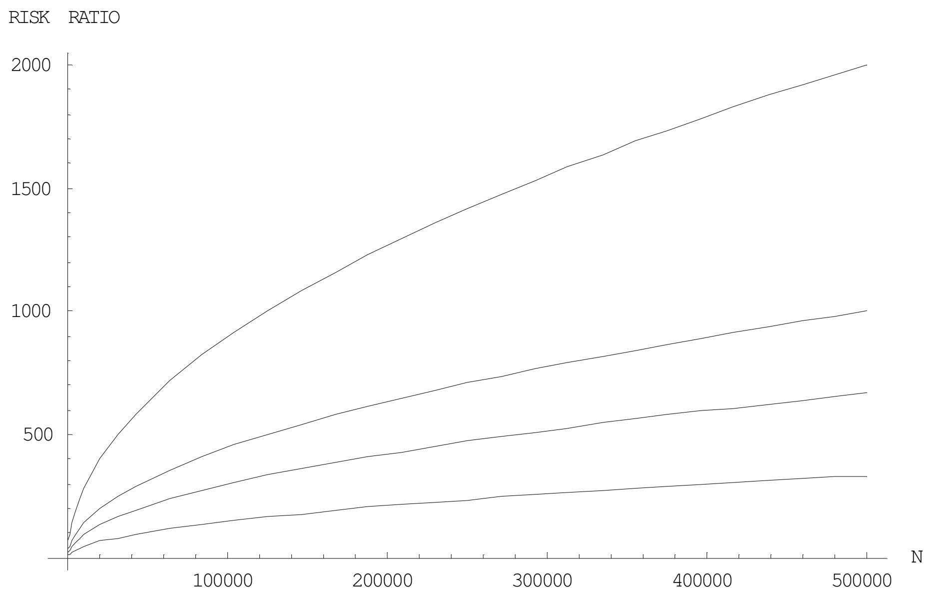

Figure 5.1: Risk Ratios a Function of the Severity Variable *n*

In figure 5.1 it can be seen that for a severity variable $n = 500.000$, or, equivalently, a probability of 10^{-6} for the type II event and losses which are an order of magnitude of 500.000 more severe than those under a type I event having a probability of 0.5, and sigma levels of $k = 1,\ 2,\ 3,\ 6$, one has risk ratios of about 334, 667, 1000, and 2000, respectively, as may be glanced from (5.22):

$$\frac{\text{Risk}_{HILP}}{\text{Risk}_{LIHP}} = \frac{2 + k\sqrt{10^6 - 1}}{3} \approx 333k \ . \tag{5.23}$$

So, the adjusted criterion of choice (5.2) – which is the location measure which takes into account the probabilistic worst case, expected, and best case scenarios (5.18) – corroborates and quantifies the common sense observation that most people judge a HILP (high impact low probability) event as more undesirable than a LIHP (low impact high probability) event, even if the expected consequence of the two events would be exactly the same.

6.0 CONCLUSION

This report contains a general decision making protocol which is inspired by Bernoulli's expected utility theory. If losses are represented by negative numbers and gains by positive numbers, then it is postulated as the most primitive decision theoretical axiom that the action which corresponds with the outcome probability distribution which falls most to the right will correspond with the most profitable action. So in the proposed decision making protocol actions are chosen on the basis of the positions of the respective outcome probability distributions of these actions.

Different measures of position are discussed (and compared) in this deliverable. Based upon that discussion (and comparison) it is recommended to take that position measures that is most "fair", as it takes into account not only the most likely scenario (i.e. the expectation value) but also the worst case (i.e. the lower bound) and best case scenarios (i.e. the upper bound).

Also, if the non-linearities which are introduced by the current asset position are to be taken explicitly into account then it is recommended to use Bernoulli's utility function as this utility function (1) has a proven track record as the Fechner- Weber law, or, equivalently, Stevens' power law, of psycho-physics that describes our sense perception and (2) admits a formal consistency derivation (given in Appendix A).

Finally, it has been demonstrated that the alternative criterion of choice which is derived in Chapter 3 may resolve the consistency problems of the expected outcome and expected utility theories when it comes to the perceived difference in riskiness between Low Impact High Probability (LIHP) and High Impact Low Probability (HILP) events which have the seem expected outcomes. Moreover, it is postulated that the disproportionality factors of cost benefit analyses are an ad hoc fix-up for the insufficiency[5] of the expectation value as a criterion of choice. The alternative criterion of choice then may lead us to a principled recommendation for the computation of disproportionality factors of the cost-benefit methodology.

[5] As an aside, the insufficiency of the expectation value as a criterion of choice has proven to be so problematic that it has given rise to the scientific field of economic behaviourism.

REFERENCES

Aczel, J. (1966) *Lectures on Functional Equations and Their Applications*. Dover Publications, Inc., New York.

Bernoulli, D. (1738) Exposition of a New Theory on the Measurement of Risk. Translated from Latin into English by Dr Louise Sommer from `Specimen Theoriae Novae de Mensura Sortis', *Commentarii Academiae Scientiarum Imperialis Petropolitanas,* Tomus V, 175-192.

Bernstein, P.L. (1998) *Against the Gods; the Remarkable Story of Risk*. John Wiley & Sons, Inc.

Cox, R.T. (1946) Probability, Frequency and Reasonable Expectation. *American Journal of Physics*, 14, 1-13,.

Davies, H. (2010) *The Financial Crises; Who is to Blame?* Polity Press, Cambridge, UK.

Fancher, R.E. (1990) *Pioneers of Psychology*. W. W. Norton and Company, London.

Fechner, G.J. (1966) *Elemente der Psychophysik, 2 vols.*; Vol. 1 translated as Elements of Psychophysics, Boring, E.G. and Howes, D.H., eds. Holt, Rinehart and Winston, New York.

Goose, M.H. (2006) Gross Disproportion, Step by step – A possible approach to evaluating additional measures at COMAH sites. *In Institution of chemical engineers symposium series*, vol. 151, p. 952. Institution of chemical engineers; 1999.

Jaynes, E.T. (2003) *Probability Theory; the Logic of Science*. Cambridge University Press.

Kahneman, D. (2011) *Thinking, Fast and Slow*. Penguin Random House, UK.

Kahneman, D. & Tversky, A. (1972) Subjective Probability: a Judgment of Representativeness. *Cognitive Psychology* 3, 430-454.

Knuth, K.H. & Skilling, J. (2010) *Foundations of Inference*, arXiv: 1008.4831v1 [math.PR].

Laplace, P.S. (1819) *Essai Philosophique sur les Probabilites*, Courcier Imprimeur, Paris.

Masin, S.C., Zudini, V., & Antonelli, M. (2009) Early Alternative Derivations of Fechner's Law. *Journal of Behavioral Sciences*, 45, 56-65.

Reith, G. (2009) Uncertain times: The notion of 'Risk' and the development of modernity. *In The earthscan reader on Risk*, Lofstedt & Boholm (Eds.), Earthscan, London, UK.

Rushton, A. (2006) *CBA, ALARP and industrial safety in the United Kingdom*. UK.

Savage, L.J. (1954) *Foundations of Statistics*. J.Wiley & Sons, NY.

Stevens, S.S. (1961) To Honor Fechner and Repeal His Law, *Science*, New Series, Vol. 133, No. 3446, 80-86.

Thomas, P. & Jones, R. (2010). Extending the j-value framework for safety analysis to include the environmental costs of a large accident. *Process Safety and Environmental Protection*, 88(5), p. 297-317.

Tversky, A. & Kahneman, D. (1992) Advances in Prospect Theory: Cumulative Representation of Uncertainty. *Journal of Risk and Uncertainty*, 5: 297-323.

Van Erp, H.RN., Linger, R.O., & Van Gelder, P.H.A.J.M. (2016*a*) An Outline of the Bayesian Decision Theory. *In Bayesian Inference and Maximum Entropy Methods in Science and Engineering-35th International Workshop*, Potsdam, New York

Van Erp, H.RN., Linger, R.O., & Van Gelder, P.H.A.J.M. (2016*b*) Bayesian Decision Theory: A Simple Toy Problem. *In Bayesian Inference and Maximum Entropy Methods in Science and Engineering-35th International Workshop*, Potsdam, New York.

Van Erp, H.R.N., Linger, R.O., & Van Gelder, P.H.A.J.M. (2015) Fact Sheet Reseach on Bayesian Decision Theory, arXiv.

Von Neumann, J. & Morgenstern, O. (1946) *Theory of Games and Economic Behavior*, Princeton University Press.

APPENDIX A: CONSISTENT DERIVATION OF BERNOUILLI'S UTILITY FUNCTION

We will now derive the Bernoulli utility function, or, equivalently, the Weber-Fechner law, or, equivalently, in content, Steven's Power law, using the desiderata of invariance and consistency. In this we follow a venerable Bayesian tradition (Cox, 1946; Jaynes, 2003; Knuth and Skilling, 2010).
Say, we have the positive quantities x, y, and z, of some stimulus or commodity of interest. Then these quantities, being numbers on the positive real, admit an ordering. So, let quantities be ordered as $x \leq y \leq z$. We now want to find the function f that quantifies the perceived decrease associated with going from, say, the quantity z to the quantity x.
The first functional equation is based on the desideratum that the unknown function f should be invariant for a change of scale in our quantities:

$$f(x,z) = f(cx,cy) \tag{A.1}$$

where c is positive constant. For example, if our quantities concern sums of money, then the perceived loss of going from ten dollars to one dollar should be the same perceived loss if we reformulate this scenario in dollar cents.
The second functional equation is based on the desideratum of consistency, in which we state that the perceived decrease in going directly from z to x, ought to be the same perceived decrease in going from z to x via y:

$$f(x,z) = g[f(x,y), f(y,z)] \tag{A.2}$$

For example, if our quantities concern sums of money, then the perceived loss of going from ten dollars to one dollar should be the same perceived loss if we first go from ten dollars to five dollars, and then from five dollars to one dollar; seeing that in both scenarios we start out with an initial wealth of ten dollars, only to end up with a current wealth of one dollar.
The general solution to (A.1) is (van Erp *et al.*, 2016*a*):

$$f(x,y) = h\left(\frac{x}{y}\right) \tag{A.3}$$

were h is some arbitrary function. The general solution to (A.2) is (Knuth and Skilling, 2010):

$$\Theta[f(x,z)] = \Theta[f(x,y)] + \Theta[f(y,z)] \tag{A.4}$$

where Θ is some arbitrary monotonic function. Moreover, because of this arbitrariness, we may define Θ as (Knuth and Skilling, 2010):

$$\Theta(u) = \log \Psi(u), \tag{A.5}$$

where Ψ itself is also arbitrary and monotonic. Using (A.5), we may rewrite (A.4), without any loss of generality, as

$$\log \Psi[f(x,z)] = \log \Psi[f(x,y)] + \log \Psi[f(y,z)] \tag{A.6}$$

or, equivalently, by exponentiation of both sides of (A.6),

$$\Psi[f(x,z)] = \Psi[f(x,y)]\, \Psi[f(y,z)] \tag{A.7}$$

Substituting (A.3) into (A.4) through (A.7), and letting, respectively,

$$\theta\left(\frac{x}{y}\right) = \Theta\left[h\left(\frac{x}{y}\right)\right] \tag{A.8}$$

and

$$\psi\left(\frac{x}{y}\right) = \Psi\left[h\left(\frac{x}{y}\right)\right] \tag{A.9}$$

we obtain the equivalent functional equations:

$$\theta\left(\frac{x}{z}\right)=\theta\left(\frac{x}{y}\right)+\theta\left(\frac{y}{z}\right) \tag{A.10}$$

and

$$\psi\left(\frac{x}{z}\right)=\psi\left(\frac{x}{y}\right)\psi\left(\frac{y}{z}\right) \tag{A.11}$$

If we assume differentiability, then (A.10), together with the two boundary conditions:

$$f(x,x)=\theta\left(\frac{x}{x}\right)=0 \tag{A.12}$$

and

$$f(x,y)=\theta\left(\frac{x}{y}\right)<0, \qquad \text{for } x<y, \tag{A.13}$$

is sufficient to find the function f that quantifies the perceived decrease associated with going from the quantity y to the quantity x. This function f turns out to be Bernoulli's utility function, or, equivalently, the Weber-Fechner law of sense perception:

$$f(x,y)=q\log\frac{x}{y}, \qquad \text{for } q>0, \tag{A.14}$$

where y is our initial asset position and x is the final asset position, and q is some arbitrary constant which has to be obtained by way psychological experimentation.

So, Bernoulli's utility function (A.14) is the only function that adheres to the desiderata of unit invariance and consistency, respectively, (A.1) and (A.2), and the boundary conditions that a zero change should lead to a zero perceived loss and that a perceived loss should be assigned a negative value, respectively, (A.12) and (A.13). Any other utility function will be in violation with these fundamental desiderata and specific boundary conditions.

Note that Fechner re-derived (A.14) in 1860 as the law that guides our sensory perception. In the years that followed (A.14) proved to be so successful, as it, amongst other things, gave rise to our decibel scale, that it established psychology as a legitimate experimental science (Fancher, 1990).

But as Fechner was very careful, for reasons of aesthetics, or so we hazard to guess (van Erp *et al.*, 2015), to apply his Weber law, which later became the Fechner-Weber law, only to non-monetary stimuli, the implied universality of (A.14) was not recognized for the longest time. However, because of the here given consistency derivation of (A.14), it is now shown that the Fechner-Weber, or, equivalently, Bernoulli's utility function, is one of the consistent functions that quantifies the distance between x and y; thus, explaining the universal applicability of Bernoulli's utility function.

The other consistent distance function is Steven's power law, which may be derived as follows. If we assume differentiability, then (A.11), together with the two boundary conditions:

$$f(x,x)=\psi\left(\frac{x}{x}\right)=1 \tag{A.15}$$

and

$$0<f(x,y)=\psi\left(\frac{x}{y}\right)<1, \qquad \text{for } x<y, \tag{A.16}$$

is sufficient to find the function f that quantifies the perceived decrease associated with going from the quantity y to the quantity x. This function f turns out to be Steven's power law:

$$f\left(x,y\right)=\left(\frac{x}{y}\right)^{q}, \qquad \text{for } q>0, \tag{A.17}$$

Where y is our initial asset position and x is the final asset position, and q is some arbitrary constant which has to be obtained by way psychological experimentation.

So, Steven's power law (A.17) is the only function that adheres to the desiderata of unit invariance and consistency, respectively, (A.1) and (A.2), and the boundary conditions that a zero change should lead to a ratio of one between the initial and final asset position and that a perceived loss should be assigned a value smaller than 1, respectively, (A.15) and (A.16). Any other utility function will be in violation with these fundamental desiderata and specific boundary conditions.

We summarize, given the desiderata (A.1) and (A.2), the Fechner-Weber law (A.14) results from the boundary condition that negative increments result negative utilities and a zero increment results in an utility of zero, (A.12) and (A.13); whereas Steven's power law (A.17) results from the boundary condition that utilities must be greater than zero and that a zero increment results an utility of one, (A.15) and (A.16). Stated differently, the Fechner-Weber law and Steven's power law are both equivalent in content, differing only in the proposed utility scale. A subtlety that seems to have been overlooked by some, seeing that the Fechner-Weber law versus the Steven's power law has been a source of controversy in psycho-physical community (Stevens, 1961).

In closing, It may be read in (Jaynes, 2003), that to the best of Jaynes' knowledge, there are as of yet no formal principles at all for assigning numerical values to loss functions; not even when the criterion is purely economic, because the utility of money remains ill-defined. In the absence of these formal principles, Jaynes final verdict was that decision theory cannot be fundamental. The Bernoulli utility function, initially derived by Bernoulli, by way of common sense first principles (Bernoulli, 1738), has now been derived by way of a consistency argument.

This consistency argument explains why it is that Bernoulli's utility function, both in its original Fechner-Weber law and in its alternative Steven's power law form, has proven to be so ubiquitous and successful the field of sensory perception research; simply because human sense perception, like the laws of Nature (Knuth, 2014*b*), adheres to the desideratum of consistency.